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ON THE GEOMETRY AND CLASSIFICATION OF ABSOLUTE PARALLELISMS. II

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8. The irreducible case

Let (M, ds^2) be a simply connected globally symmetric pseudo-riemannian manifold, and ϕ an absolute parallelism on M consistent with ds^2. We assume (M, ds^2) to be irreducible. Our standing notation is

$\mathfrak{p}$: the LTS of ϕ-parallel vector fields on M,

$\mathfrak{g}$: the Lie algebra of all Killing vector fields on M,

σ_x: conjugation of $\mathfrak{g}$ by the symmetry s_x at $x \in M$,

$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$: eigenspace decomposition under σ_x.

The irreducibility says that $\mathfrak{m}$ is a simple noncommutative LTS, and thus (Lemma 6.2) says the same for $\mathfrak{p}$.

8.1. Lemma. *Either* $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}$ *or* $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} = 0$.

Proof. Let $\mathfrak{i} = [\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p}$. Then $[[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}] \subset \mathfrak{p}$ implies $[\mathfrak{i}, \mathfrak{p}] \subset \mathfrak{i}$ and so $[\mathfrak{i}\mathfrak{p}\mathfrak{p}] \subset \mathfrak{i}$. Thus $\mathfrak{i}$ is a LTS ideal in $\mathfrak{p}$. By simplicity, either $\mathfrak{i} = 0$ or $\mathfrak{i} = \mathfrak{p}$.

If $\mathfrak{i} = 0$, then $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} = 0$. If $\mathfrak{i} = \mathfrak{p}$, then $\mathfrak{p} \subset [\mathfrak{p}, \mathfrak{p}]$. As $[\mathfrak{i}, \mathfrak{p}] \subset \mathfrak{i}$, also $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$. Hence $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}$. q.e.d.

We do the group manifolds immediately.

8.2. Proposition. *Let (M, ds^2) be irreducible simply connected and globally symmetric, with consistent absolute parallelism ϕ such that the* LTS *of ϕ-parallel fields satisfies $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} \neq 0$. Then $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}$, $\mathfrak{p}$ is a simple real Lie algebra, and $(M, \phi, ds^2) \cong (P, \lambda, d\sigma^2)$ where*

(i) *P is the simply conncted group for $\mathfrak{p}$,*

(ii) *λ is the parallelism of left translation on P, and*

(iii) *$d\sigma^2$ is the bi-invariant metric induced by a nonzero multiple of the Killing form of $\mathfrak{p}$.*

The symmetry of $(P, d\sigma^2)$ at $1 \in P$ is given by $s(x) = x^{-1}$. The group G of all isometries of $(P, d\sigma^2)$ has isotropy subgroup K at 1 given by

$$K = \mathrm{Aut}_R(\mathfrak{p}) \cup s \cdot \mathrm{Aut}_R(\mathfrak{p}) .$$

The identity component G_0 of G is locally isomorphic to $P \times P$, acting by left and right translations. G is the disjoint union of cosets $\alpha \cdot G_0$ and $s\alpha \cdot G_0$ as α

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runs through a system of representatives of $\mathrm{Aut}_R(\mathfrak{p})/\mathrm{Int}(\mathfrak{p})$. *Finally,* $s(\lambda)$ *is the parallelism of right translation, and is the only other absolute parallelism on* P *consistent with* $d\sigma^2$.

Proof. Theorem 3.8, Lemma 8.1, fact (10.6), and the fact that any invariant bilinear form on a real simple Lie algebra is a multiple of the Killing form, give us $(M, \phi, ds^2) \cong (P, \lambda, d\sigma^2)$ with $s(\lambda) = \rho$, as claimed. The assertions on G and K follow from (5.2) and the fact that every derivation of a simple Lie algebra is inner. q.e.d.

Now we start in on the non-group case.

8.3. Lemma. *Let* $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} = 0$. *Then* $\mathfrak{g}$ *is simple,* $\mathfrak{g} = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$, *and there is an automorphism*

$$(8.4) \qquad \varepsilon_x : \mathfrak{g} \to \mathfrak{g} \quad \textit{such that} \quad \varepsilon_x(\xi) = \xi - \sigma_x(\xi) \quad \textit{for } \xi \in \mathfrak{p} .$$

Proof. $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$ is faithfully represented as the Lie algebra of all LTS derivations of $\mathfrak{m}$. Now (10.3) shows $\mathfrak{g} = \mathfrak{l}_S(\mathfrak{m})$ standard Lie enveloping algebra; as $\mathfrak{m}$ is simple this forces $\mathfrak{g} = \mathfrak{l}_U(\mathfrak{m})$ universal Lie enveloping algebra. If $\mathfrak{g}$ were not simple, then (10.7) $\mathfrak{m}$ would be the LTS of a Lie algebra, and Theorem 3.8 would force $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$. Thus $\mathfrak{g}$ is simple.

Let $h: \mathfrak{m} \to \mathfrak{p}$ be the inverse of the LTS isomorphism f_x of Lemma 6.2. Then h extends to a Lie algebra homomorphism of $\mathfrak{l}_U(\mathfrak{m}) = \mathfrak{g}$ onto the algebra $[\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$ generated by $\mathfrak{p}$. As $\mathfrak{g}$ is simple, $h: \mathfrak{g} \cong [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$. In particular $[\mathfrak{p}, \mathfrak{p}] + \mathfrak{p} = \mathfrak{g}$ and we realize ε_x as h^{-1}. q.e.d.

Our method consists of showing that σ_x and ε_x generate such a large group of outer automorphisms of $\mathfrak{g}$ that we can deduce $\mathfrak{g}$ to be of type D_4 and ε_x to be the triality. Some technical problem (proving σ_x outer) forces us to reduce to the compact case.

We construct a compact riemannian version of (M, ds^2). Choose

$$(8.5a) \qquad \theta: \quad \text{Cartan involution of } \mathfrak{g} .$$

Thus θ is an involutive automorphism of $\mathfrak{g}$, whose fixed point set is a maximal compactly embdded subalgebra $\mathfrak{l} \subset \mathfrak{g}$. Let $\mathfrak{q}$ be the -1 eigenspace of θ on $\mathfrak{g}$. Then we have

$$(8.5b) \qquad \mathfrak{g} = \mathfrak{l} + \mathfrak{q} \quad \text{Cartan decomposition under } \theta .$$

Now choose $x \in M$ so that σ_x commutes with θ. That is always possible because the σ_z, $z \in M$, form a conjugacy class of semi-simple automorphisms of $\mathfrak{g}$. That done, we have

$$(8.5c) \qquad \mathfrak{k} = (\mathfrak{k} \cap \mathfrak{l}) + (\mathfrak{k} \cap \mathfrak{q}) , \qquad \mathfrak{m} = (\mathfrak{m} \cap \mathfrak{l}) + (\mathfrak{m} \cap \mathfrak{q}) .$$

Now define

(8.6a) $\mathfrak{g}^* = \mathfrak{l} + i\mathfrak{q}$ compact real form of $\mathfrak{g}^C$,

and define subspaces of $\mathfrak{g}^*$ by

(8.6b) $\mathfrak{k}^* = \mathfrak{k}^C \cap \mathfrak{g}^*$, $\mathfrak{m}^* = \mathfrak{m}^C \cap \mathfrak{g}^*$.

σ_x extends to $\mathfrak{g}^C$ by linearity and then restricts to an automorphism (still denoted σ_x) of $\mathfrak{g}^*$. Now

(8.6c) $\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{m}^*$ eigenspace decomposition under σ_x .

To pass to the group level we define
G^*: simply connected group with Lie algebra $\mathfrak{g}^*$,
K^*: analytic subgroup for $\mathfrak{k}^*$.
Then G^* is a compact semisimple group, and K^* is a closed subgroup because it is identity component of the fixed point set of σ_x on G^*. Now we have

$$M^* = G^*/K^*: \quad \text{compact simply connected manifold.}$$

The Killing form κ of $\mathfrak{g}^*$ is negative definite, so the restriction of $-\kappa$ to $\mathfrak{m}^*$ induces
dt^2: G^*-invariant riemannian metric on M^*.
We summarize the main properties as follows.

8.7. Lemma. *(M^*, dt^2) is a simply connected globally symmetric riemannian manifold of compact type, and $\mathfrak{g}^*$ is the Lie algebra of all Killing vector fields on (M^*, dt^2). For simple $\mathfrak{g}$, (M^*, dt^2) is irreducible if and only if $\mathfrak{g}^C$ is simple. If $\mathfrak{g}$ is simple but $\mathfrak{g}^C$ is not simple, then $\mathfrak{g} = \mathfrak{l}^C$ with $\mathfrak{l}$ compact simple and σ_x C-linear on $\mathfrak{g}$, and $\mathfrak{g}^* = \mathfrak{l} \oplus \mathfrak{l}$ with $\mathfrak{k}^* = (\mathfrak{k} \cap \mathfrak{l}) \oplus (\mathfrak{k} \cap \mathfrak{l})$.*

Proof. The riemannian metric dt^2 is symmetric because it is induced by an invariant bilinear form $-\kappa$ of $\mathfrak{g}^*$. As $\mathfrak{g}^*$ is semisimple and σ_x-stable it must contain every Killing vector field of (M^*, dt^2).

If $\mathfrak{g}^C$ is simple, then $\mathfrak{g}^*$ is simple, so (M, dt^2) is irreducible. If (M, dt^2) irreducible, then $\mathfrak{m}^*$ is a simple LTS; if further $\mathfrak{g}$ is simple, then $\mathfrak{m}$ (thus also $\mathfrak{m}^*$) is not the LTS of a Lie algebra; thus $\mathfrak{g}^*$ is simple, and that proves $\mathfrak{g}^C$ simple.

Suppose $\mathfrak{g}$ to be simple but $\mathfrak{g}^C$ not simple. Then $\mathfrak{g} = \mathfrak{l}^C$ where the maximal compactly embedded subalgebra $\mathfrak{l}$ is a compact real form. To avoid confusion we write $\mathfrak{g} = \mathfrak{l} + j\mathfrak{l}$ with $j^2 = -1$. Were σ_x antilinear on $\mathfrak{g}$ its fixed point set $\mathfrak{k}$ would be a real form, so $\mathfrak{g} = \mathfrak{k} + j\mathfrak{k}$ and $\mathfrak{m} = j\mathfrak{k}$; then $\mathfrak{k}$ would be absolutely irreducible on $\mathfrak{m}$, so (M, dt^2) would be irreducible, contradicting nonsimplicity of $\mathfrak{g}^C$. Thus σ_x is complex-linear on $\mathfrak{g}$. Now the fixed point set $\mathfrak{k} = (\mathfrak{k} \cap \mathfrak{l})^C$, and the assertions on $\mathfrak{g}^*$ and $\mathfrak{k}^*$ follow. q.e.d.

If (M, ds^2) is compact, then $(M^*, dt^2) = (M, cds^2)$ for some real $c \neq 0$. If (M, ds^2) is riemannian, then (Corollary 4.5) it is compact.

We carry ϕ over to an absolute parallelism on (M^*, dt^2).

8.8. Lemma. *The Cartan involution θ can be chosen so that $\theta(\mathfrak{p}) = \mathfrak{p}$. Assume θ so chosen, and define $\mathfrak{p}^* = \mathfrak{p}^C \cap \mathfrak{g}^*$. Then there is an absolute parallelism ϕ^* on M^* consistent with dt^2, such that $\mathfrak{p}^*$ is the LTS of ϕ^*-parallel vector fields on M^*. If $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}$, then $[\mathfrak{p}^*, \mathfrak{p}^*] = \mathfrak{p}^*$. If $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} = 0$, then $[\mathfrak{p}^*, \mathfrak{p}^*] \cap \mathfrak{p}^* = 0$.*

Proof. If $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}$, then $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{v}$ with each summand stable under any choice of θ, and $\mathfrak{p} = \mathfrak{v} \oplus 0$. Then $\mathfrak{g}^* = \mathfrak{v}^* \oplus \mathfrak{v}^*$ with $\mathfrak{p}^* = \mathfrak{v}^* \oplus 0$ and all the assertions are trivial.

Now suppose $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} = 0$. Then from (8.4) we have an involutive automorphism $\pi = \varepsilon_x^{-1}\sigma_x\varepsilon_x$ whose fixed point set is $[\mathfrak{p}, \mathfrak{p}]$ and whose -1 eigenspace is $\mathfrak{p}$. Note that this shows π to be independent of x. As π is a semisimple automorphism of $\mathfrak{g}$, we can choose θ to commute with π.

We now assume further that θ commutes with π. In other words, using (8.5),

$$(8.9a) \quad [\mathfrak{p}, \mathfrak{p}] = ([\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{l}) + ([\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{q}) , \qquad \mathfrak{p} = (\mathfrak{p} \cap \mathfrak{l}) + (\mathfrak{p} \cap \mathfrak{q}) .$$

From this we see

$$(8.9b) \qquad [\mathfrak{p}^*, \mathfrak{p}^*] = [\mathfrak{p}, \mathfrak{p}]^C \cap \mathfrak{g}^* , \quad \text{so} \quad \mathfrak{g}^* = [\mathfrak{p}^*, \mathfrak{p}^*] + \mathfrak{p}^* .$$

In order to proceed we must check that

$$(8.10) \qquad (1 - \sigma_x)[\mathfrak{p}, \mathfrak{p}] = \mathfrak{m} , \qquad (1 - \sigma_x)[\mathfrak{p}^*, \mathfrak{p}^*] = \mathfrak{m}^* .$$

In view of (8.9) it suffices to check the first of these assertions. If $(1 - \sigma_x)[\mathfrak{p}, \mathfrak{p}] \neq \mathfrak{m}$, then we have $0 \neq u \in \mathfrak{m}$ such that

$$b_x((1 - \sigma_x)[\xi, \eta], u) = 0 \qquad \text{for all } \xi, \eta \in \mathfrak{p} .$$

Let $\zeta \in \mathfrak{p}$ with $(1 - \sigma_x)\zeta = u$. Now

$$ds_x^2(\xi, [\eta, \zeta]) = ds_x^2([\xi, \eta], \zeta) = 0 \qquad \text{for all } \xi, \eta \in \mathfrak{p}$$

implying $[\mathfrak{p}, \zeta] = 0$. Applying ε_x now $[\mathfrak{m}, u] = 0$. As $\mathfrak{m}$ is a simple noncommutative LTS now $u = 0$. We conclude $(1 - \sigma_x)[\mathfrak{p}, \mathfrak{p}] = \mathfrak{m}$, and (8.10) is verified.

Let J^* denote the analytic subgroup of G^* for $[\mathfrak{p}^*, \mathfrak{p}^*]$. It is closed in G^*, thus compact, because it is the identity component of the fixed point set of the automorphism $\pi = \varepsilon_x^{-1}\sigma_x\varepsilon_x$ on G^*. Denote

$$(8.11a) \qquad x^* = 1 \cdot K^* \in M^* .$$

Now (8.10) shows $J^*(x^*)$ is open in M^*. As J^* is compact, so is $J^*(x^*)$. Thus

$$(8.11b) \qquad J^*(x^*) = M^* .$$

Recall that dt^2 is induced by negative of the Killing form κ of $\mathfrak{g}^*$. Note that $\frac{1}{2}(1-\sigma_x)$ is κ-orthogonal projection of $\mathfrak{g}^*$ to $\mathfrak{m}^*$, and also from (8.9) that ε_x is well defined on $\mathfrak{g}^*$. Now let $\xi, \eta \in \mathfrak{p}^*$. If $j \in J^*$, then $\text{ad}(j)^{-1}\xi$, $\text{ad}(j)^{-1}\eta \in \mathfrak{p}^*$, and we compute

$$\begin{aligned} 4dt^2_{j(x^*)}(\xi,\eta) &= 4dt^2_{x^*}(\text{ad}(j)^{-1}\xi,\ \text{ad}(j)^{-1}\eta) \\ &= -\kappa((1-\sigma_x)\,\text{ad}(j)^{-1}\xi,\ (1-\sigma_x)\,\text{ad}(j)^{-1}\eta) \\ &= -\kappa(\varepsilon_x\,\text{ad}(j)^{-1}\xi,\ \varepsilon_x\,\text{ad}(j)^{-1}\eta) = -\kappa(\xi,\eta)\ , \end{aligned}$$

which is independent of the choice of $j \in J^*$. But (8.11) says that every element of M^* is of the form $j(x^*)$. Thus

(8.12) if $\xi, \eta \in \mathfrak{p}^*$, then $dt^2(\xi,\eta)$ is constant on M^* .

Choose a basis $\{\xi_1, \cdots, \xi_n\}$ of $\mathfrak{p}^*$. The ξ_{ix^*} form a basis of $M^*_{x^*}$ because $(1-\sigma_x)\mathfrak{p}^* = \mathfrak{m}^*$. Now (8.12) says that $\{\xi_1, \cdots, \xi_n\}$ is a global frame on M^* with the $dt^2(\xi_i, \xi_j)$ constant. Recall that the ξ_i are Killing vector fields of (M^*, dt^2). Corollary 4.15 now says that M^* has an absolute parallelism ϕ^* consistent with dt^2 such that $\mathfrak{p}^*$ is the space of ϕ^*-parallel vector fields. q.e.d.

If $\mathfrak{l}$ is a Lie algebra over a field F, then $\text{Aut}_F(\mathfrak{l})$ denotes the group of all automorphisms of $\mathfrak{l}$ over F. If $F = R$ or $F = C$, then $\text{Int}(\mathfrak{l})$ denotes the normal subgroup of $\text{Aut}_F(\mathfrak{l})$ consisting of inner automorphisms, i.e., generated by the $\exp(\text{ad}\, v)$ with $v \in \mathfrak{l}$. If $\mathfrak{l}$ is real or complex semisimple, then $\text{Int}(\mathfrak{l})$ is the identity component of the Lie group $\text{Aut}_F(\mathfrak{l})$.

Now we begin to identify (M, ds^2).

8.13. Lemma. *Suppose* $[\mathfrak{p},\mathfrak{p}] \cap \mathfrak{p} = 0$. *If* $\alpha \in \text{Aut}_R(\mathfrak{g})$ *is induced by an isometry of* (M, ds^2), *in particular, if* $\alpha \in \text{Int}(\mathfrak{g})$, *then* $\alpha(\mathfrak{m}) \neq \mathfrak{p}$, *and* $\varepsilon_x\alpha$ *does not commute with* σ_x. *If* $\alpha^* \in \text{Aut}_R(\mathfrak{g}^*)$ *is induced by an isometry of* (M^*, dt^2), *in particular, if* $\alpha^* \in \text{Int}(\mathfrak{g}^*)$, *then* $\alpha^*(\mathfrak{m}^*) \neq \mathfrak{p}^*$, *and* $\varepsilon_x\alpha^*$ *does not commute with* σ_x.

Proof. Let $\alpha \in \text{Aut}_R(\mathfrak{g})$ induced by an isometry a of (M, ds^2). Then $\psi = a^{-1}(\phi)$ is an absolute parallelism on M consistent with ds^2, and the LTS of ψ-parallel vector fields is $\alpha^{-1}(\mathfrak{p})$. If $\alpha(\mathfrak{m}) = \mathfrak{p}$, then $\mathfrak{m}$ is the LTS of ψ-parallel fields, and the comparison of (4.7) with (5.2) proves (M, ds^2) to be flat. As (M, ds^2) is not flat, we conclude $\alpha(\mathfrak{m}) \neq \mathfrak{p}$. In particular, $\varepsilon_x\alpha(\mathfrak{m}) \neq \mathfrak{m}$, i.e., $\varepsilon_x\alpha$ does not preserve the -1 eigenspace of σ_x, so $\varepsilon_x\alpha$ does not commute with σ_x.

Lemma 8.8 allows us to use the same argument for α^*, $\mathfrak{m}^*$ and $\mathfrak{p}^*$. q.e.d.

If $\mathfrak{g}^C$ is not simple, Lemma 8.7 tells us $\mathfrak{g} = \mathfrak{l}^C$ where $\mathfrak{l}$ is compact simple and $\sigma_x \in \text{Aut}_C(\mathfrak{l}^C)$. However, it is conceivable that our extension $\varepsilon_x \in \text{Aut}_R(\mathfrak{g})$ of $f_x : \mathfrak{p} \cong \mathfrak{m}$ be complex antilinear. Should that be the case, note that the Cartan involution θ is complex antilinear on $\mathfrak{l}^C$, so $\varepsilon_x\theta \in \text{Aut}_C(\mathfrak{l}^C)$. Thus either

(8.14a) $\varepsilon_x \in \mathrm{Aut}_C(\mathfrak{l}^C)$ and we denote $\varepsilon'_x = \varepsilon_x \in \mathrm{Aut}_C(\mathfrak{l}^C)$,

or

(8.14b) $\varepsilon_x \notin \mathrm{Aut}_C(\mathfrak{l}^C)$ and we denote $\varepsilon'_x = \varepsilon_x\theta \in \mathrm{Aut}_C(\mathfrak{l}^C)$.

8.15. Lemma. *Let* $[\mathfrak{p},\mathfrak{p}] \cap \mathfrak{p} = 0$. *If* $\mathfrak{g}^C$ *is simple, then* $\mathrm{Int}(\mathfrak{g}^C), \sigma_x \cdot \mathrm{Int}(\mathfrak{g}^C)$ *and* $\varepsilon_x \cdot \mathrm{Int}(\mathfrak{g}^C)$ *are three distinct components of* $\mathrm{Aut}_C(\mathfrak{g}^C)$. *If* $\mathfrak{g}^C$ *is not simple, so* $\mathfrak{g} = \mathfrak{l}^C$ *with* $\mathfrak{l}$ *compact simple, then* $\mathrm{Int}(\mathfrak{g})$, $\sigma_x \cdot \mathrm{Int}(\mathfrak{g})$ *and* $\varepsilon'_x \cdot \mathrm{Int}(\mathfrak{g})$ *are three distinct components of* $\mathrm{Aut}_C(\mathfrak{l}^C)$.

Proof. First consider the case where $\mathfrak{g}^C$ is simple. Then $\mathfrak{g}^*$ is simple and (M^*, dt^2) is irreducible. Every nonzero element of $\mathfrak{p}^*$ is a never-vanishing vector field on M^*, so the Euler-Poincaré characteristic $\chi(M^*) = 0$. That implies rank $G^* >$ rank K^*, so σ_x is an outer automorphism on $\mathfrak{g}^*$. Now $\sigma_x \notin \mathrm{Int}(\mathfrak{g}^C)$.

If ε_x is an inner automorphism of $\mathfrak{g}^C$, then it is inner on $\mathfrak{g}^*$ giving $\alpha^* = \varepsilon_x^{-1} \in \mathrm{Int}(\mathfrak{g}^*)$ such that $\varepsilon_x\alpha^*$ commutes with σ_x. Thus Lemma 8.13 forces $\varepsilon_x \notin \mathrm{Int}(\mathfrak{g}^C)$.

It σ_x and ε_x differ by an inner automorphism of $\mathfrak{g}^C$, then $\alpha^* = \varepsilon_x^{-1}\sigma_x \in \mathrm{Int}(\mathfrak{g}^*)$ such that $\varepsilon_x\alpha^*$ commutes with σ_x. Thus Lemma 8.13 forces $\sigma_x \cdot \mathrm{Int}(\mathfrak{g}^C) \cap \varepsilon_x \cdot \mathrm{Int}(\mathfrak{g}^C)$ to be empty.

The assertions are proved for $\mathfrak{g}^C$ simple. Now suppose $\mathfrak{g}^C$ to be not simple. Then $\mathfrak{g} = \mathfrak{l}^C$ with $\mathfrak{l}$ compact simple and $\sigma_x \in \mathrm{Aut}_C(\mathfrak{l}^C)$ by Lemma 8.7, and we have $\varepsilon'_x \in \mathrm{Aut}_C(\mathfrak{l}^C)$ as in (8.14). Now $\mathfrak{g}^* \cong \mathfrak{l} \oplus \mathfrak{l}$ with each summand stable under σ_x, so the argument for simple $\mathfrak{g}^C$ shows σ_x to be outer on each summand of $\mathfrak{g}^*$. It follows that σ_x is outer on $\mathfrak{l}^C = \mathfrak{g}$, i.e., that $\sigma_x \notin \mathrm{Int}(\mathfrak{g})$.

If ε'_x is inner on $\mathfrak{l}^C$ then $\alpha' = \varepsilon_x'^{-1} \in \mathrm{Int}(\mathfrak{g})$ and $\varepsilon'_x\alpha'$ commutes with σ_x. From (8.5c) we see that θ is induced by an isometry of (M, ds^2). Thus $\varepsilon_x\alpha$ commutes with σ_x, where either $\alpha = \alpha'$ or $\alpha = \theta\alpha'$, and where α is induced by an isometry of (M, ds^2). That contradicts Lemma 8.13, forcing $\varepsilon'_x \notin \mathrm{Int}(\mathfrak{g})$. A similar modification of the argument for simple $\mathfrak{g}^C$ proves $\sigma_x \cdot \mathrm{Int}(\mathfrak{g}) \cap \varepsilon'_x \cdot \mathrm{Int}(\mathfrak{g})$ to be empty.

The assertions are proved for $\mathfrak{g}^C$ non-simple. q.e.d.

Given integers p, $q \geq 0$ and a basis $\{e_1, \cdots, e_{p+q}\}$ of R^{p+q} we have the symmetric nondegenerate bilinear form $b_{p,q}$ on R^{p+q} given by

$$b_{p,q}\left(\sum_{i=1}^{p+q} a^i e_i, \sum_{j=1}^{p+q} c^j e_j\right) = \sum_{k=1}^{p} a^k c^k - \sum_{k=1}^{q} a^{p+k} c^{p+k} .$$

Now denote

$$\mathrm{O}(p, q): \quad \text{real orthogonal group of } b_{p,q} ,$$

so the usual orthogonal group in m real variables is $\mathrm{O}(m) = \mathrm{O}(m, 0)$. Now $\mathrm{O}(p, q)$ has four components if $pq \neq 0$, and two components if $pq = 0$. Denote

$SO(p,q)$: identity component of $O(p,q)$,
$\mathfrak{so}(p,q)$: Lie algebra of $O(p,q)$.
Then of course

$$SO(m) = SO(m,0) \ , \qquad \mathfrak{so}(m) = \mathfrak{so}(m,0) \ .$$

Consider the $(p+q-1)$-manifold

$$SO(p,q)(e_1) \cong SO(p,q)/SO(p-1,q) \ , \qquad p \geq 1 \ ;$$

$b_{p,q}$ induces a pseudo-riemannian metric of signature $(p-1,q)$ and constant curvature 1 under which it is globally symmetric, and the case $q = 0$ is the sphere $S^{p-1} = SO(p)/SO(p-1)$. We also have

$$SO(p,q)(e_{p+q}) \cong SO(p,q)/SO(p,q-1) \ , \qquad q \geq 1 \ ;$$

there $b_{p,q}$ induces a globally symmetric preudo-riemannian metric of signature $(p,q-1)$ and constant curvature -1, and the case $q = 1$ is the real hyperbolic space $H^p = SO(p,1)/SO(p)$. Finally denote

$$O(m,C) = O(m)^C \quad \text{complex orthogonal group of } b_{p,m-p} \ ;$$

$$SO(m,C) = SO(m)^C \quad \text{identity component; and}$$

$$\mathfrak{so}(m,C) = \mathfrak{so}(m)^C \quad \text{Lie algebra of } SO(m,C) \ .$$

Viewing $R^{p+q} \subset C^{p+q}$ we have $(m = p + q)$

$$SO(m,C)(e_i) \cong SO(m,C)/SO(m-1,C) \ ,$$

globally symmetric pseudo-riemannian manifold of signature $(m-1, m-1)$ and nonconstant curvature, affine complexification of S^{m-1}.

Finally we have our classification. Recall that we are using the notation
G: group of all isometries of (M, ds^2);
$\mathfrak{g}$: Lie algebra of G, Killing fields of (M, ds^2);
$x \in M$ and $K = \{g \in G \colon g(x) = x\}$ so $M = G/K$;
$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$: decomposition under symmetry σ_x;
$\mathfrak{p}$: the LTS of ϕ-parallel vector fields on M.

8.16. Theorem. *Let (M, ds^2) be an irreducible simply connected globally symmetric pseudo-riemannian manifold with consistent absolute parallelism ϕ. If $[\mathfrak{p},\mathfrak{p}] \cap \mathfrak{p} \neq 0$, then (M, ϕ, ds^2) is a group manifold as in Proposition* 8.2. *If $[\mathfrak{p},\mathfrak{p}] \cap \mathfrak{p} = 0$, then there are just three cases, all of which occur, as follows.*

Case 1. *$M = SO(8)/SO(7)$, the sphere S^7, and ds^2 is a positive or negative multiple of the $SO(8)$-invariant riemannian metric of constant curvature* 1. *Here $G = O(8)$ and $K = O(7)$, 2-component groups.*

Case 2. $M = SO(4,4)/SO(3,4)$, *diffeomorphic to* $S^3 \times R^4$, *and* ds^2 *is a positive or negative multiple of the* $SO(4,4)$*-invariant pseudo-riemannian metric of signature* $(3,4)$ *and constant curvature* 1. *Here* $G = O(4,4)$ *and* $K = O(3,4)$, 4-*component groups.*

Case 3. $M = SO(8, C)/SO(7, C)$, *affine complexification of* S^7 *and diffeomorphic to* $S^7 \times R^7$, *and* ds^2 *is a multiple of the nonconstant curvature metric of signature* $(7,7)$ *induced by the Killing form of* $SO(8, C)$. *Here*

$$G = O(8, C) \cup \nu \cdot O(8, C)\ , \qquad K = O(7, C) \cup \nu \cdot O(7, C)\ ,$$

where ν *is complex conjugation of* C^8 *over* R^8 (*so that conjugation by* ν *is a Cartan involution* θ *of* G_0).

All possibilities for ϕ *are as follows. There is a triality automorphism* ε *of order* 3 *on* $\mathfrak{g}$ *with fixed point set* $\mathfrak{g}^\varepsilon$ *of type* G_2 *such that both* ε *and* σ_x *commute with a Cartan involution* θ. *Denote*

$$\mathfrak{p}_0 = \varepsilon^{-1}(\mathfrak{m}) \quad \textit{so that} \quad [\mathfrak{p}_0, \mathfrak{p}_0] = \varepsilon^{-1}(\mathfrak{k})\ ,$$

and observe that

$$\varepsilon^{-1}(\mathfrak{k}) \textit{ is the image of the spin representation of } \mathfrak{k}\ .$$

Denote

$$J = \{j \in G\colon \mathrm{ad}\,(j)\mathfrak{p}_0 = \mathfrak{p}_0\}\ , \quad \textit{and} \quad \mathfrak{p}_r = \mathrm{ad}\,(g)\mathfrak{p}_0 \textit{ for } r = gJ \in G/J\ .$$

Then J_0 *is the analytic subgroup of* G *for* $\varepsilon^{-1}(\mathfrak{k})$, *and*

(i) $J = \{\pm I_8\} \cdot J_0$ 2-*component group in cases* 1 *and* 2, $J = \{\pm I_8, \pm \nu\} \cdot J_0$ 4-*component group in case* 3;

(ii) *the* $\mathfrak{p}_r$, $r \in G/J$, *are mutually inequivalent under the action of* G;

(iii) *if* $r \in G/J$ *then there is an absolute parallelism* ϕ_r *on* M *consistent with* ds^2 *whose LTS is* $\mathfrak{p}_r$;

(iv) *every absolute parallelism on* M *consistent with* ds^2 *is in the* 7-*parameter*[4] *family* $\{\phi_r\}_{r \in G/J}$;

(v) *the parameter space* G/J *of* $\{\phi_r\}$ *is diffeomorphic* (via ε) *to the disjoint union of two copies of* $M/\{\pm I_8\}$; *and*

(vi) J_0 *is transitive on* M.

Proof. If $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} \neq 0$, we apply Proposition 8.2. Now suppose $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} = 0$.

First, consider the case where $\mathfrak{g}$ is a compact simple Lie algebra. Then $\mathfrak{g}^C$ is simple and Lemma 8.15 says that $\mathrm{Aut}_C\,(\mathfrak{g}^C)/\mathrm{Int}\,(\mathfrak{g}^C)$ has order ≥ 3, so $\mathrm{Aut}_R\,(\mathfrak{g})/\mathrm{Int}\,(\mathfrak{g})$ has order ≥ 3. This implies that $\mathfrak{g}$ is of Cartan classification type D_4, i.e., $\mathfrak{g} = \mathfrak{so}(8)$. Again by Lemma 8.15, ε_x is triality, and σ_x is outer

[4] The parameters are real in cases 1 and 2, and complex in case 3.

on $\mathfrak{g}$, so the possibilities for $\mathfrak{k}$ are $\mathfrak{so}(7)$ and $\mathfrak{so}(3) \oplus \mathfrak{so}(5)$. In the latter case $\mathfrak{k}$ and $\varepsilon_x(\mathfrak{k})$ would be $\mathrm{Int}(\mathfrak{g})$-conjugate, so we would have $\alpha \in \mathrm{Int}(\mathfrak{g})$ with $\varepsilon_x \alpha(\mathfrak{k}) = \mathfrak{k}$; then $\varepsilon_x \alpha$ commutes with σ_x in violation of Lemma 8.13. Thus $\mathfrak{k} = \mathfrak{so}(7)$ and $M = SO(8)/SO(7) = S^7$, as in case 1. Invariance forces ds^2 to be a multiple of the standard riemannian metric $d\sigma^2$ of constant curvature 1. Then $(M, d\sigma^2)$ and (M, ds^2) have the same isometry group, so $G = O(8)$, whence $K = O(7)$.

Second, consider the case where $\mathfrak{g}$ is noncompact but $\mathfrak{g}^C$ is simple. Then $\mathfrak{g}^*$ is simple. Lemma 8.8 and the argument for compact simple $\mathfrak{g}$ show that $\mathfrak{g}^* = \mathfrak{so}(8)$, $\mathfrak{k}^* = \mathfrak{so}(7)$ and $M^* = S^7$, and that ε_x is triality on $\mathfrak{g}^*$. The noncompact real forms of $\mathfrak{so}(8, C)$ are the $\mathfrak{so}(p, 8-p)$, $1 \leq p \leq 4$; the real form $\mathfrak{so}^*(8)$ whose maximal compactly embedded subalgebra is the Lie algebra $\mathfrak{u}(4)$ of the unitary group in four complex variables, is triality-equivalent to $\mathfrak{so}(2,6)$. However $\mathfrak{g}$ is stable under the triality automorphism ε_x of $\mathfrak{g}^C = \mathfrak{so}(8, C)$. Let $Y = G_0/L$, irreducible symmetric space of noncompact type where L is a maximal compact subgroup of G_0; now ε_x induces an isometry e of Y. Let $e = ab$ where $a \in G_0$ and $b(1 \cdot L) = 1 \cdot L$; then conjugation by b induces an automorphism β of $\mathfrak{l}$ which extends to a triality automorphism of $\mathfrak{g}$, so β^2 is an outer automorphism of $\mathfrak{l}$. If β is an automorphism of $\mathfrak{so}(7)$, of $\mathfrak{so}(2) \oplus \mathfrak{so}(6)$, or of $\mathfrak{so}(3) \oplus \mathfrak{so}(5)$, then β^2 is inner. We conclude that $\mathfrak{g} = \mathfrak{so}(4,4)$, which in fact does admit triality from the split Cayley algebra. Thus $\mathfrak{k} = \mathfrak{so}(3,4)$, $M = SO(4,4)/SO(3,4)$, and ds^2, G and K are specified as in case 2.

Third, consider the case where $\mathfrak{g}^C$ is not simple. Lemma 8.7 says $\mathfrak{g} = \mathfrak{l}^C$ with $\mathfrak{l}$ compact simple, $\mathfrak{k} = (\mathfrak{k} \cap \mathfrak{l})^C$, $\mathfrak{g}^* = \mathfrak{l} \oplus \mathfrak{l}$ and $\mathfrak{k}^* = (\mathfrak{k} \cap \mathfrak{l}) \oplus (\mathfrak{k} \cap \mathfrak{l})$. The argument for compact simple $\mathfrak{g}$ says $\mathfrak{l} = \mathfrak{so}(8)$, $\mathfrak{k} \cap \mathfrak{l} = \mathfrak{so}(7)$ and $M^* = S^7 \times S^7$. Thus $\mathfrak{g} = \mathfrak{so}(8, C)$, $\mathfrak{k} = \mathfrak{so}(7, C)$ and $M = SO(8, C)/SO(7, C)$. Now ds^2, G and K are specified as in case 3.

It remains to verify the assertions on the construction of all consistent absolute parallelisms for the spaces (M, ds^2) of cases 1, 2 and 3.

Let $M = G/K$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ as in case 1, 2 or 3 of the theorem. Then $\mathfrak{g}$ admits a triality automorphism ε of order 3 with fixed point set $\mathfrak{g}^\varepsilon$ of type G_2 [12, Table 7.14]. Fix a Cartan involution θ of $\mathfrak{g}$ which commutes with σ_x. As $\varepsilon^3 = 1$, ε is a semisimple automorphism of $\mathfrak{g}$, so we may replace ε by an $\mathrm{Int}(\mathfrak{g})$-conjugate if necessary to arrange $\varepsilon\theta = \theta\varepsilon$. That done we use θ to construct a compact real form $\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{m}^*$ of $\mathfrak{g}^C$ as in (8.5) and (8.6), and ε extends by linearity to $\mathfrak{g}^C$ preserving $\mathfrak{g}^*$. Define $\mathfrak{p}_0 = \varepsilon^{-1}(\mathfrak{m})$ as prescribed; then $\mathfrak{p}_0^* = \mathfrak{p}_0^C \cap \mathfrak{g}^*$ is $\varepsilon^{-1}(\mathfrak{m}^*)$.

Let κ denote the Killing form on $\mathfrak{g}$. We need to prove the following facts:

(8.17a) $\qquad (1 - \sigma_x)\mathfrak{p}_0 = \mathfrak{m}\ , \qquad (1 - \sigma_x)[\mathfrak{p}_0, \mathfrak{p}_0] = \mathfrak{m}\ ,$ and

(8.17b) $\qquad$ if $\ \xi, \eta \in \mathfrak{p}_0\ ,\ $ then $\ \kappa(\xi, \eta) = \kappa((1 - \sigma_x)\xi, (1 - \sigma_x)\eta)\ .$

To do this we note that $\mathfrak{g}^\varepsilon = \mathfrak{k} \cap \varepsilon^{-1}(\mathfrak{k})$, so the orthocomplement of $\mathfrak{g}^\varepsilon$ in $\mathfrak{g}$

relative to κ is $\mathfrak{k}^{\perp} + \varepsilon^{-1}(\mathfrak{k}^{\perp}) = \mathfrak{m} + \varepsilon^{-1}(\mathfrak{m}) = \mathfrak{m} + \mathfrak{p}_0$. Now ε^{-1} is a rotation by $2\pi/3$ on $\mathfrak{m}^* + \mathfrak{p}_0^*$. As $\frac{1}{2}(1 - \sigma_x)$ is the orthogonal projection of $\mathfrak{m}^* + \mathfrak{p}_0^*$ to $\mathfrak{m}^*$, that says $\kappa(\xi, \eta) = \kappa((1 - \sigma_x)\xi, (1 - \sigma_x)\eta)$ for $\xi, \eta \in \mathfrak{p}_0^*$. The same follows by linearity for $\xi, \eta \in \mathfrak{p}_0^C$, and thus for $\xi, \eta \in \mathfrak{p}_0$. That proves (8.17b), and the first assertion of (8.17a) follows. Let dim denote $\dim_R$ in cases 1 and 2, and $\dim_C$ in case 3. Then $\dim \mathfrak{g} = 28$, $\dim \mathfrak{k} = 21$, $\dim \mathfrak{g}' = 14$ and $\dim \mathfrak{m} = 7$. Thus $\dim (1 - \sigma_x)[\mathfrak{p}_0, \mathfrak{p}_0] = \dim \varepsilon^{-1}(\mathfrak{k}) - \dim \mathfrak{g}' = 21 - 14 = 7 = \dim \mathfrak{m}$, proving the second part of (8.17a). Now (8.17) is verified.

As prescribed, let J be the normalizer of $\mathfrak{p}_0$ in G. As $\mathfrak{k}$ is the normalizer of $\mathfrak{m}$ in $\mathfrak{g}$, so $[\mathfrak{p}_0, \mathfrak{p}_0] = \varepsilon^{-1}(\mathfrak{k})$ is the Lie algebra of J, and assertion (i) on the structure of J follows.

Let $j \in J$ and $\xi, \eta \in \mathfrak{p}_0$, and let β be the multiple of κ that induces ds^2. We compute

$$\begin{aligned} 4ds^2_{j(x)}(\xi, \eta) &= 4ds^2_x(\mathrm{ad}\,(j)^{-1}\xi, \mathrm{ad}\,(j)^{-1}\eta) \\ &= 4\beta(\tfrac{1}{2}(1 - \sigma_x)\,\mathrm{ad}\,(j)^{-1}\xi, \tfrac{1}{2}(1 - \sigma_x)\,\mathrm{ad}\,(j)^{-1}\eta) \\ &= \beta((1 - \sigma_x)\,\mathrm{ad}\,(j)^{-1}\xi, (1 - \sigma_x)\,\mathrm{ad}\,(j)^{-1}\eta) \\ &= \beta(\mathrm{ad}\,(j)^{-1}\xi, \mathrm{ad}\,(j)^{-1}\eta) = \beta(\xi, \eta)\ , \end{aligned}$$

which is independent of $j \in J$. Thus $ds^2(\xi, \eta)$ is constant on $J(x)$. However (8.17a) says that the Lie algebra $[\mathfrak{p}_0, \mathfrak{p}_0]$ of J orthogonally projects onto $\mathfrak{m}$. Thus $J(x)$ is open in M. Now choose a basis $\{\xi_1, \cdots, \xi_n\}$ of $\mathfrak{p}_0$. We have just checked that the $ds^2(\xi_i, \xi_j)$ are constant on the open set $J(x) \subset M$. Now $(1 - \sigma_x)\mathfrak{p}_0 = \mathfrak{m}$ shows that $\{\xi_1, \cdots, \xi_n\}$ is a global frame on $J(x)$. Thus Corollary 4.15 says that there is an absolute parallelism ψ on the connected manifold $J_0(x)$, consistent with ds^2 there, for which the ξ_i are parallel. Lemma 6.4 says that (M, ds^2) has an absolute parallelism ϕ_0 such that the $\xi|_{J_0(x)}$, $\xi \in \mathfrak{p}_0$, are ϕ_0-parallel on $J_0(x)$. By analyticity, or because ϕ_0-parallel fields are Killing vector fields, now $\mathfrak{p}_0$ is the LTS of all ϕ_0-parallel vector fields on M.

If $r = gJ \in G/J$, we define $\mathfrak{p}_r = \mathrm{ad}\,(g)\mathfrak{p}_0$ as specified. Then $\phi_r = g(\phi_0)$ is an absolute parallelism on M consistent with ds^2, and its LTS is $\mathrm{ad}\,(g)\mathfrak{p}_0 = \mathfrak{p}_r$. This gives us our 7-parameter family $\{\phi_r\}$ of absolute parallelisms consistent with ds^2.

We check that the original absolute parallelism ϕ on M is contained in the family $\{\phi_r\}$. Let Aut $(\mathfrak{g})$ denote $\mathrm{Aut}_R\,(\mathfrak{g})$ in cases 1 and 2, and $\mathrm{Aut}_C\,(\mathfrak{g})$ in case 3. Then Aut $(\mathfrak{g})/$Int $(\mathfrak{g})$ is the group of order 6 given by $e^3 = s^2 = 1$, $ses^{-1} = e^{-1}$. Here s represents the component of σ_x, and e the component of ε. Thus ε_x (or ε'_x in case 3) is in a component represented by e, es, ses^{-1} or se. Now there are isometries $g, b \in G$ of (M, ds^2) such that $\varepsilon_x = \mathrm{ad}\,(b)\cdot\varepsilon\cdot\mathrm{ad}\,(g)^{-1}$ and either $b = 1$ or $b = s_x$ symmetry. Let $r = gJ \in G/J$. Then $\mathfrak{p} = \varepsilon_x^{-1}(\mathfrak{m}) = \mathrm{ad}\,(g)\cdot\varepsilon^{-1}\cdot\mathrm{ad}\,(b^{-1})(\mathfrak{m}) = \mathrm{ad}\,(g)\varepsilon^{-1}(\mathfrak{m}) = \mathrm{ad}\,(g)\mathfrak{p}_0 = \mathfrak{p}_r$. Thus $\phi = \phi_r$.

Assertion (ii) on the structure of J and $\{\phi_r\}$ is immediate from the definition of J. We have just proved assertions (iii) and (iv). Now (i), (v) and (vi) remain.

Let $N = G_0/J_0$, and let β be the multiple of the Killing form of $\mathfrak{g}$ which induces ds^2 on M. Then β induces a metric du^2 on N, and ε induces an isometry of (N, du^2) onto (M, ds^2). If $g \in G$, we notice that $\mathrm{ad}(g)^2$ is an inner automorphism of $\mathfrak{g}$. If h is an isometry of (N, du^2), it follows that $ad(h)^2$ is an inner automorphism of $\mathfrak{g}$. Thus $\mathfrak{p}_0 \neq \mathrm{ad}\,(g)\varepsilon^{-1}(\mathfrak{p}_0)$ whenever $g \in G_0$, for $(\mathrm{ad}\,(g)\varepsilon^{-1})^2$ is outer on $\mathfrak{g}$. If J meets $s_x G_0$, say $gs_x \in J$ where $g \in G_0$, then

$$\begin{aligned}\mathfrak{p}_0 &= \mathrm{ad}\,(g)\sigma_x(\mathfrak{p}_0) = \mathrm{ad}\,(g)\sigma_x\varepsilon^{-1}(\mathfrak{m}) = \mathrm{ad}\,(g)\varepsilon\sigma_x(\mathfrak{m}) \\ &= \mathrm{ad}\,(g)\varepsilon(\mathfrak{m}) = \mathrm{ad}\,(g)\varepsilon^2(\mathfrak{p}_0) = \mathrm{ad}\,(g)\varepsilon^{-1}(\mathfrak{p}_0)\ ,\end{aligned}$$

which was just seen impossible. Thus

(8.18a) J does not meet the component $s_x G_0$ of G .

The Int ($\mathfrak{g}$)-normalizer of $\mathfrak{m}$ is the connected group $\mathrm{ad}\,(K_0 \cup (-I_8)K_0)$, so the normalizer of $\mathfrak{p}_0 = \varepsilon^{-1}(\mathfrak{m})$ in Int ($\mathfrak{g}$) is $\mathrm{ad}\,(J_0 \cup (-I_8)J_0)$. Thus

(8.18b) $$J \cap G_0 = \begin{cases}\{\pm I_8\}\cdot J_0 \ \text{(2 components) in cases 1 and 3,} \\ J_0 \ \text{(connected) in case 2.}\end{cases}$$

Note $\nu \in J$ in case 3. Denote

$$J' = \{\pm I_8\}\cdot J_0 \ \text{in cases 1 and 2, and} \qquad J' = \{\pm I_8, \pm\nu\}\cdot J_0 \ \text{in case 3.}$$

J' meets one of the two components of G in case 1, and meets two of the four components of G in cases 2 and 3. Thus $G/J'G_0$ has order 2. But (8.18a) says that G/JG_0 has order ≥ 2. As $J' \subset J$, now $JG_0 = J'G_0$. However, (8.18b) says $J \cap G_0 = J' \cap G_0$. We conclude $J = J'$, thus proving assertion (i) on the structure of J.

In view of (i), G/J is the disjoint union of two copies of $G_0/(J \cap G_0) = G_0/\{\pm I_8\}\cdot J_0$. Since the isometry $(N, du^2) \to (M, ds^2)$ induced by ε, where $N = G_0/J_0$, induces a diffeomorphism of $G_0/\{\pm I_8\}\cdot J_0$ onto $M/\{\pm I_8\}$. Assertion (v) follows.

Recall that the Lie algebra $\varepsilon^{-1}(\mathfrak{k})$ of J is the image of the spin representation of $\mathfrak{k}$. Thus

(8.19a) $J_0 = \mathrm{Spin}\,(7), \mathrm{Spin}\,(3, 4), \mathrm{Spin}\,(7, C)$ in cases 1, 2, 3.

Recall also that $\mathfrak{k} \cap \varepsilon^{-1}(\mathfrak{k}) = \mathfrak{g}^\varepsilon$ algebra of type G_2. Let G_2 denote the compact connected group of that type, G_2^C the complex connected group of that type, and $G_2^\sharp$ the analytic subgroup of G_2^C which is the noncompact real form. Now

(8.19b) $(J \cap K)_0 = G_2, G_2^\sharp, G_2^C$ in cases 1, 2, 3 .

Now count dimensions, or recall from (8.17a), to see that

$$J_0(x) \text{ is open in } M.$$

In case 1, where J_0 is compact, this give us $J_0(x) = M$.

In cases 2 and 3, we choose a basis $\{e_1, \cdots, e_8\}$ of the ambient space R^8 or C^8 of M such that the e_k are mutually orthogonal, each $\|e_k\|^2 = |b(e_k, e_k)| = 1$, and

case 2: $U = e_1R + e_2R + e_3R + e_4R$ is positive definite, and
$V = e_5R + e_6R + e_7R + e_8R$ is negative definite;

case 3: $U = e_1R + \cdots + e_8R$ is positive definite, and so
$V = iU = ie_1R + \cdots + ie_8R$ is negative definite.

Then

$$e_1 \in M = \{u + v : u \in U, v \in V \quad \text{and} \quad \|u\|^2 - \|v\|^2 = 1\} .$$

Given real $r > s \geq 0$ with $r^2 - s^2 = 1$ we define

$$S_{r,s} = \{u + v : u \in U, v \in V, \|u\|^2 = r^2 \quad \text{and} \quad \|v\|^2 = s^2\} .$$

Now M is the disjoint union of the $S_{r,s}$.

As J_0 is noncompact semisimple, its Lie algebra has an element $w \neq 0$ which is diagonable with all eigenvalues real. The eigenvalues come in pairs $\{h, -h\}$ by (8.19a). Renormalizing w, now we may assume $\{e_1, \cdots, e_8\}$ chosen so that

case 2: $w(e_1 + e_5) = e_1 + e_5$ and $w(e_1 - e_5) = -(e_1 - e_5)$;
case 3: $w(e_1 + ie_2) = e_1 + ie_2$ and $w(e_1 - ie_2) = -(e_1 - ie_2)$.

Now by direct calculation

$$\exp(tw) \cdot e_1 \in S_{\cosh(t), \sinh(t)} , \qquad t \geq 0 .$$

Thus $J_0(e_1)$ meets each of the sets $S_{r,s}$.

Let $H = \{g \in J_0 : g(U) = U\}$. Then also $g(V) = V$ for $g \in H$, and H is the maximal compact subgroup

$$\text{Spin}(3) \cdot \text{Spin}(4) \text{ in case 2,} \qquad \text{Spin}(7) \text{ in case 3.}$$

In case 2 the Spin (3)-factor on H is transitive on the sphere $\|u\|^2 = r^2$ in U, and the Spin (4)-factor is transitive on the sphere $\|v\|^2 = s^2$ in V. Thus H is transitive on each $S_{r,s}$. As $J_0(e_1)$ meets each $S_{r,s}$, now $J_0(e_1) = M$.

In case 3, H is transitive on the sphere $\|u\|^2 = r^2$ in U, and the subgroup H_1 preserving e_1 is G_2 by (8.19b). Thus H_1 is transitive on the spheres $\|v_1\|^2 = s_1^2$ in $i(e_2R + e_3R + \cdots + e_8R)$. If $z \in S_{r,s}$, then some element of H carries z to $z' = re_1 + i(ae_1 + be_2)$ where $b \geq 0$ and $a^2 + b^2 = s^2$. However, $z' \in M$ says

$(r + ia)^2 + (ib)^2 = 1$ so $ra = 0$; as $r > 0$ now $a = 0$; thus $z' = re_1 + ise_2$. Choose $t \geq 0$ such that $r = \cosh(t)$, so $s = \sinh(t)$; now

$$z' = \cosh(t)e_1 + i\sinh(t)e_2 = \exp(tw)\cdot e_1 \ .$$

Thus $J_0(e_1) = M$, and (vi) is proved, completing the proof of Theorem 8.16.

9. Global classification of reductive parallelisms

Theorems 7.6 and 8.16 completely describe the possibilities for the (M_i, ϕ_i, ds_i^2) in Theorem 6.7. Splitting the flat factor as in the proof of Proposition 7.5, we thus reformulate Theorem 6.7 as follows.

9.1. Theorem. *Let (M, ϕ, ds^2) be a connected manifold with absolute parallelism and consistent pseudo-riemannian metric such that ϕ is of reductive type relative to ds^2. Then there exist*

(1) *unique integers $t \geq u \geq 0$,*

(2) *simply connected globally symmetric pseudo-riemannian manifolds (M_i, ds_i^2), $-1 \leq i \leq t$, unique up to global isometry aud permutations of $\{1, 2, \cdots, u\}$ and $\{u + 1, u + 2, \cdots, t\}$, and*

(3) *absolute parallelisms ϕ_i on M_i consistent with ds_i^2 and unique up to global isometry, such that the (M_i, ϕ_i, ds_i^2) and*

$$(\tilde{M}, \tilde{\phi}, d\sigma^2) = (M_{-1}, \phi_{-1}, ds_{-1}^2) \times \cdots \times (M_t, \phi_t, ds_t^2)$$

have the following properties:

(i) *For $-1 \leq i \leq u$, M_i is the simply connected group for a real Lie algebra $\mathfrak{p}_i$, ϕ_i is its absolute parallelism of left translation, and ds_i^2 is the bi-invariant metric induced by a nondegenerate invariant bilinear form b_i on $\mathfrak{p}_i$. Here $(\mathfrak{p}_{-1}, b_{-1})$ is obtained as in (7.2) and (7.4a), and $\mathfrak{p}_{-1}$ has center $\mathfrak{z}_{-1} = \mathfrak{z}_{-1}^{\perp}$ relative to b_{-1}; so (M_{-1}, ds_{-1}^2) is flat. $\mathfrak{p}_0$ is commutative, so (M_0, ds_0^2) is flat and ϕ_0 is its euclidean parallelism. If $1 \leq i \leq u$, then $\mathfrak{p}_i$ is simple and b_i is a nonzero real multiple of its real Killing form, so (M_i, ds_i^2) is irreducible.*

(ii) *For $u + 1 \leq i \leq t, M_i$ is one of the symmetric coset spaces G_0/K_0 given by*

$SO(8)/SO(7)$	*ordinary*	*7-sphere,*
$SO(4,4)/SO(3,4)$	*indefinite*	*7-sphere, or*
$SO(8,C)/SO(7,C)$	*complexified*	*7-sphere;*

ds_i^2 is induced by a nonzero real multiple of the real Killing form of G_0, and ϕ_i comes from a triality automorphism of $\mathfrak{g}$ as in Theorem 8.16.

(iii) *Every $x \in M$ has a neighborhood U and an isometry $h: (U, ds^2) \to (\tilde{U}, d\sigma^2)$, $\tilde{U}$ open in $\tilde{M}$, such that h sends $\phi|_U$ to $\tilde{\phi}|_{\tilde{U}}$.*

(iv) *If ϕ is complete,* i.e., *if (M, ds^2) is complete, then there is a pseudo-riemannian covering $\pi: (\tilde{M}, d\sigma^2) \to (M, ds^2)$ which sends $\tilde{\phi}$ to ϕ.*

We draw two corollaries of Theorems 3.8, 7.6 and 8.16 which complement the statement of Theorem 9.1.

9.2. Corollary. *Let* $(\tilde{M}, \tilde{\phi}, d\sigma^2)$ *be a complete simply connected pseudo-riemannian manifold with consistent absolute parallelism of reductive type.*

(i) *Then the group of all isometries* g *of* $(\tilde{M}, d\sigma^2)$ *such that* $g(\tilde{\phi}) = \tilde{\phi}$ *is transitive on* $\tilde{M}$.

(ii) *If* $(\tilde{M}, d\sigma^2)$ *has no euclidean (flat) factor, and* $\tilde{\psi}$ *is another absolute parallelism consistent with* $d\sigma^2$, *then* $(\tilde{M}, d\sigma^2)$ *has an isometry* g *such that* $(g\tilde{\psi}) = \tilde{\phi}$.

Proof. $(\tilde{M}, \tilde{\phi}, d\sigma^2)$ is the product of the (M_i, ϕ_i, ds_i^2), $-1 \leq i \leq t$, as in Theorem 9.1. If $-1 \leq i \leq u$ there, then the left translations of the group manifold M_i are transitive and preserve ϕ_i. If $u + 1 \leq i \leq t$, then the required transitivity is the transitivity of the group J in Theorem 8.16. Thus (i) holds for each (M_i, ϕ_i, ds_i^2), and thus for $(\tilde{M}, \tilde{\phi}, d\sigma^2)$. Similarly, (ii) follows from Proposition 8.2 and Theorem 8.16.

9.3. Corollary. *Let* ds^2 *be of signature* $(n - q, q)$ *or* $(q, n - q)$, $0 \leq q \leq 2$, *in Theorem* 9.1.

(i) M_{-1} *is reduced to a point*, i.e., *the parallelism on the flat factor of* $(\tilde{M}, d\sigma^2)$ *is euclidean.*

(ii) *At most* q *of the simple group manifolds* $M_i\,(1 \leq i \leq u)$ *are noncompact. Each noncompact one is the universal covering group of* $SL(2, R)$.

(iii) *Each of the quadrics* M_i $(u + 1 \leq i \leq t)$ *is an ordinary 7-sphere.*

(iv) *If* $\tilde{\psi}$ *is any absolute parallelism on* $\tilde{M}$ *consistent with* $d\sigma^2$, *then* $(\tilde{M}, d\sigma^2)$ *has an isometry* g *such that* $g(\tilde{\psi}) = \tilde{\phi}$.

Proof. If M_{-1} is not reduced to a point, then $\mathfrak{p}_{-1}$ is nonabelian by the normalization $\mathfrak{z}_{-1} = \mathfrak{z}_{-1}^{\perp}$ (rel. b_{-1}) of Theorem 9.1 (i). Then the 3-form τ in the construction (7.2) of $\mathfrak{p}_{-1}$ must be nonzero. But τ is a 3-form on an r-dimensional vector space where ds_{-1}^2 has signature (r, r). The latter implies $r \leq 2$ so $\tau = 0$. Assertion (i) follows.

Let the simple group manifold M_i $(1 \leq i \leq u)$ be noncompact, and $\mathfrak{p}_i = \mathfrak{l}_i + \mathfrak{q}_i$ the decomposition of its Lie algebra under a Cartan involution. If $l_i = \dim \mathfrak{l}_i$ and $q_i = \dim \mathfrak{q}_i$, then ds_i^2 has signature (l_i, q_i) or (q_i, l_i). Thus either $l_i \leq 2$ or $q_i \leq 2$. If $l_i \leq 2$, then $\mathfrak{l}_i$ has no simple ideal, so $\mathfrak{l}_i$ is 1-dimensional by simplicity of $\mathfrak{p}_i$; then R-irreducibility of $\mathfrak{l}_i$ on $\mathfrak{q}_i$ implies $q_i \leq 2$. If $q_i \leq 2$, the symmetric space of noncompact type associated to $\mathfrak{p}_i$ must have constant curvature and therefore must be the real hyperbolic plane, so $\mathfrak{p}_i$ is the Lie algebra of $SL(2, R)$. Each such M_i contributes $(1, 2)$ or $(2, 1)$ to the signature of ds^2, so at most q occur. Assertion (ii) is proved.

The quadrice M_i $(u + 1 \leq i \leq t)$ have ds_i^2 of signature

$$
\begin{array}{lccc}
SO(8)/SO(7): & (7,0) & \text{or} & (0,7)\ ; \\
SO(4,4)/SO(3/4): & (3,4) & \text{or} & (4,3)\ ; \\
SO(8,C)/SO(7,C): & & & (7,7)\ .
\end{array}
$$

The last two quadrics are excluded because $q < 3$. That leaves the 7-sphere, proving assertion (iii).

Let $\tilde{\psi}$ be another absolute parallelism on $\tilde{M}$ consistent with $d\sigma^2$. Then $\tilde{\psi}$ is of reductive type by Lemma 6.2, and assertion (i) for $(\tilde{M}, \tilde{\psi}, d\sigma^2)$ shows $\tilde{\psi}$ is euclidean on the flat factor of $(\tilde{M}, d\sigma^2)$. Thus Lemma 6.2 shows $(\tilde{M}, \tilde{\psi}, d\sigma^2)$ to be the product of the (M_i, ψ_i, ds_i^2) for certain ψ_i with $\psi_0 = \phi_0$. Now assertion (iv) follows from Corollary 9.2. q.e.d.

Our goal now is a complete description of the possibilities for the coverings of Theorem 9.1 (4).

9.4. Lemma. *Let $\pi\colon (M', d\sigma^2) \to (M, ds^2)$ be a pseudo-riemannian covering, and ϕ an absolute parallelism on M consistent with ds^2. Let $\mathfrak{p}$ be the* LTS *of ϕ-parallel vector fields on M, and $\mathfrak{p}'$ the space of all fields ξ' on M' with $\pi_*\xi'$ defined and in $\mathfrak{p}$.*

(i) *There is a unique absolute parallelism ϕ' on M' such that $\pi(\phi') = \phi$. It is consistent with $d\sigma^2$, and $\mathfrak{p}'$ is its* LTS *of parallel vector fields.*

(ii) *If $\xi' \in \mathfrak{p}'$ and γ is a deck transformation of the covering, then $\gamma_*\xi' = \xi'$.*

Proof. Assertion (i) is immediate with ϕ' defined by the condition that $\mathfrak{p}'$ be its LTS. Then $\pi_*\colon \mathfrak{p}' \cong \mathfrak{p}$, so as $\pi \circ \gamma = \pi$ implies $\pi_*\gamma_*\xi' = \pi_*\xi'$ we get $\gamma_*\xi' = \xi'$.

9.5. Proposition. *Let $(M', \phi', d\sigma^2)$ be a connected pseudo-riemannian manifold with consistent absolute parallelism, and Z be the Lie group of all isometries g of $(M', d\sigma^2)$ such that if ξ' is ϕ'-parallel then $g_*\xi' = \xi'$.*

(i) *If $1 \neq g \in Z$, then g has no fixed point on M'.*

(ii) *A subgroup of Z is discrete if, and only if, it acts freely and properly discontinuously on M'.*

(iii) *The normal pseudo-riemannian coverings $\pi\colon (M', d\sigma^2) \to (M, ds^2)$ such that $\pi(\phi')$ is a well-defined absolute parallelism on M are just the coverings $M' \to D\backslash M'$ where D is a discrete subgroup of Z.*

Proof. Let $g \in Z$ have a fixed point $x \in M'$. The tangent space M'_x consists of all ξ'_x with ξ' a ϕ'-parallel vector field. As each $g_*\xi' = \xi'$ now $g_*\colon M'_x \to M'_x$ identity map. Since g is an isometry and M' is connected, this shows $g = 1$, and hence (i) is proved.

Choose a basis $\{\xi'_1, \cdots, \xi'_n\}$ of the space $\mathfrak{p}'$ of parallel fields. Let $\{\theta^i\}$ be the dual 1-forms. If $g \in Z$ each $g^*\theta^i = \theta^i$, so g is an isometry of the riemannian metric $d\rho^2 = \Sigma(\theta^i)^2$. The topology on Z is the compact-open topology from its action on M'. Thus a subgroup $D \subset Z$ is discrete if and only if it acts properly discontinuously on M'; it acts freely by (i). Hence (ii) is proved.

If $\pi(\phi') = \phi$ absolute parallelism on M, then ϕ is consistent with ds^2 and we are in the situation of Lemma 9.4. The covering being normal, $M = D\backslash M'$ where D is a group of homeomorphisms acting freely and properly discontinuously on M'. The elements of D are isometries of $(M', d\sigma^2)$ because π is pseudo-riemannian. Now $D \subset Z$ by Lemma 9.4, and D is discrete there by (ii). Conversely let $D \subset Z$ discrete subgroup. Then D acts freely and properly discontinuously on M' by (ii), so $\pi: M' \to D\backslash M' = M$ is a normal covering. Since D acts by isometries, π is pseudo-riemannian and $\pi(\phi')$ is a well-defined parallelism by definition of Z. Hence (iii) is proved. q.e.d.

We collect the specific information needed to apply Proposition 9.5 in the complete reductive case.

9.6. Lemma. *Let $(\tilde{M}, \tilde{\phi}, d\sigma^2)$ be a simply connected manifold with complete absolute parallelism of reductive type and consistent pseudo-riemannian metric. Let $Z(\tilde{M}, \tilde{\phi}, d\sigma^2)$ denote the Lie group of all isometries of $(\tilde{M}, d\sigma^2)$ which preserve every $\tilde{\phi}$-parallel vector field. Decompose $(\tilde{M}, \tilde{\phi}, d\sigma^2)$ as the product of the (M_i, ϕ_i, ds_i^2), as in Theorem 9.1.*

(i) *$Z(\tilde{M}, \tilde{\phi}, d\sigma^2)$ is the product of the $Z(M_i, \phi_i, ds_i^2)$.*

(ii) *If M_i is a group manifold* (i.e., $-1 \leq i \leq u$), *then $Z(M_i, \phi_i, ds_i^2)$ is its group of left translations.*

(iii) *If M_i is a quadric* (i.e., $u + 1 \leq i \leq t$), *then $Z(M_i, \phi_i, ds_i^2) = \{\pm I_8\}$.*

Proof. Let $g \in Z(\tilde{M}, \tilde{\phi}, d\sigma^2)$. Then g acts trivially on $\tilde{\mathfrak{p}} = \mathfrak{p}_{-1} \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_t$, so it preserves each ideal $\mathfrak{p}_i$. Thus $g = g_{-1} \times g_0 \times \cdots \times g_t$ where $g_i \in Z(M_i, \phi_i, ds_i^2)$, and (i) is proved.

Let M_i be a group manifold, and L_i the group of its left translations. Then $L_i \subset Z(M_i, \phi_i, ds_i^2)$. If $g \in Z(M_i, \phi_i, ds_i^2)$, we have $h \in L_i$ such that $hg(1) = 1$. Since hg is an isometry and acts trivially on $\mathfrak{p}_i$, $hg = 1$, and thus $g = h^{-1} \in L_i$, proving (ii).

Let M_i be a quadric. Then the group G_i of all isometries of (M_i, ds_i^2) has Lie algebra $\mathfrak{g}_i = [\mathfrak{p}_i, \mathfrak{p}_i] + \mathfrak{p}_i$. Let $g \in Z(M_i, \phi_i, ds_i^2)$ and $\gamma = \mathrm{ad}\,(g) \in \mathrm{Aut}_R(\mathfrak{g}_i)$. Then γ is trivial on $\mathfrak{p}_i$, and hence also trivial on $[\mathfrak{p}_i, \mathfrak{p}_i]$, so $\gamma = 1$. Now g centralizes the identity component of G_i. A glance at Theorem 8.16 shows that this forces $g = \pm I_8$, proving (iii). q.e.d.

Now we combine Theorem 9.1, Proposition 9.5 and Lemma 9.6, obtaining the classification of complete parallelisms of reductive type.

9.7. Theorem. *The complete connected pseudo-riemannian manifolds with consistent absolute parallelism of reductive type are precisely the (M, ϕ, ds^2) constructed as follows.*

Step 1. $(M_{-1}, \phi_{-1}, ds_{-1}^2)$. *Choose an integer $r \geq 0$, a real vector space $\mathfrak{w}$ of dimension r, and an alternating trilinear form $\tau \in \Lambda^3(\mathfrak{w}^*)$ which is nondegenerate on $\mathfrak{w}$ in the sense that if $0 \neq w \in \mathfrak{w}$, then $\tau(w, \mathfrak{w}, \mathfrak{w}) \neq 0$. Let $\mathfrak{p}_{-1} = \mathfrak{g}(\tau, \mathfrak{w})$ as in construction* (7.2). *Let b_{-1} be the nondegenerate invariant bilinear form* (7.4a) *on $\mathfrak{p}_{-1}$. M_{-1} is the simply connected Lie group for $\mathfrak{p}_{-1}$, ϕ_{-1} is its parallelism of left translation, and ds_{-1}^2 is the bi-invariant metric induced by b_{-1}. Note that ds_{-1}^2 has signature $(p_{-1}, q_{-1}) = (r, r)$. Let Z_{-1} denote the group of left translations on M_{-1}.*

Step 2. (M_0, ϕ_0, ds_0^2). *Choose integers $p_0, q_0 \geq 0$. M_0 is the real vector group of dimension $p_0 + q_0$, ϕ_0 is its* (*euclidean*) *parallelism of* (*left*) *translation, and ds_0^2 is a translation-invariant metric of signature (p_0, q_0). Let Z_0 denote the group of all translations.*

Step 3. *The* (M_i, ϕ_i, ds_i^2) *for* $1 \leq i \leq u$. *Choose an integer* $u \geq 0$. *If* $1 \leq i \leq u$, *let* $\mathfrak{p}_i$ *be a simple real Lie algebra,* M_i *the simply connected group for* $\mathfrak{p}_i$, ϕ_i *its parallelism of left translation, and* ds_i^2 *the bi-invariant metric induced by a nonzero real multiple of the Killing form of* $\mathfrak{p}_i$. *Let* (p_i, q_i) *denote the signature of* ds_i^2, *and* Z_i *the group of left translations of* M_i.

Step 4. *The* (M_i, ϕ_i, ds_i^2) *for* $u + 1 \leq i \leq t$. *Choose an integer* $t \geq u$. *If* $u + 1 \leq i \leq t$, *let* $M_i = G_i^0/K_i^0$ *be one of*

$$SO(8)/SO(7)\ , \quad SO(4,4)/SO(3,4)\ , \quad SO(8,C)/SO(7,C)\ .$$

ds_i^2 *is the invariant metric induced by a nonzero real multiple of the real Killing form of the Lie algebra* $\mathfrak{g}_i$ *of* G_i^0. *Let* σ *be the conjugation of* $\mathfrak{g}_i$ *by the symmetry at* $1 \cdot K_i^0$, θ *a Cartan involution of* $\mathfrak{g}_i$ *which commutes with* σ, *and* ε *a triality automorphism of order* 3 *on* $\mathfrak{g}_i$ *which commutes with* θ *and has a fixed point set of type* G_2. *Then* ϕ_i *is the absolute parallelism on* M_i *whose* LTS *is* $\mathfrak{p}_i = \{\varepsilon^{-1}(v) : v \in \mathfrak{g}_i$ *and* $\sigma(v) = -v\}$. *Let* (p_i, q_i) *denote the signature of* ds_i^2, *and* Z_i *the center* $\{\pm I_8\}$ *of the isometry group of* (M_i, ds_i^2).

Step 5. $(\tilde{M}, \tilde{\phi}, d\sigma^2)$. *Define* $\tilde{M} = M_{-1} \times M_0 \times \cdots \times M_t$, $\tilde{\phi} = \phi_{-1} \times \phi_0 \times \cdots \times \phi_t$ *and* $d\sigma^2 = ds_{-1}^2 \times ds_0^2 \times \cdots \times ds_t^2$. *Let* $p = \sum p_i$ and $q = \sum q_i$; *then* $d\sigma^2$ *has signature* (p, q). *Denote* $Z = Z_{-1} \times Z_0 \times \cdots \times Z_t$.

Step 6. $(M, \phi, ds^2) = D\backslash(\tilde{M}, \tilde{\phi}, d\sigma^2)$. *Let* $D \subset Z$ *be a discrete subgroup,* $M = D\backslash\tilde{M}$ *quotient manifold,* ϕ *parallelism on* M *induced by* $\tilde{\phi}$, *and* ds^2 *the consistent pseudo-riemannian metric of signature* (p, q) *on* M *induced by* $d\sigma^2$.

We close by examining the conditions on (M, ϕ, ds^2) under which (M, ds^2) may be globally symmetric, compact, riemannian, etc. Note that homogeneity is automatic: if (M, ϕ, ds^2) is complete and connected, then every ϕ-parallel vector field integrates to a 1-parameter group of isometries, and those isometries generate a transitive group.

9.8. Corollary. *The connected globally symmetric pseudo-riemannian manifolds with consistent absolute parallelism of reductive type are precisely the* (M, ϕ, ds^2) *constructed in Theorem* 9.7 *with the additional condition: for* $-1 \leq i \leq u$ *the projection of* D *to* Z_i *consists of translations by elements of the center of the group* M_i.

Remark. Here note that M_{-1} has center $\exp(\mathfrak{w}^*)$, that M_0 is commutative, and that M_i has discrete center for $1 \leq i \leq u$.

Proof. Let $(M, \phi, ds^2) = D\backslash(\tilde{M}, \tilde{\phi}, d\sigma^2)$ in the notation of Theorem 9.7. Then (M, ds^2) is symmetric if, and only if, every symmetry s_x of $(\tilde{M}, d\sigma^2)$ induces a transformation of M. Thus the symmetry condition for (M, ds^2) is that every s_x permute the D-orbits, i.e., that every s_x normalize D in the isometry group of $(\tilde{M}, d\sigma^2)$. Let D_i be the projection of $D \subset Z = Z_{-1} \times \cdots \times Z_t$ to Z_i. Then (M, ds^2) is symmetric if, and only if, each D_i is normalized by every symmetry of (M_i, ds_i^2).

If $u + 1 \leq i \leq t$, then $Z_i = \{\pm I_8\}$, center of the isometry group of (M_i, ds_i^2), so D_i is centralized by every symmetry.

Let $-1 \leq i \leq u$. If $x, g \in M_i$, then the symmetry of (M_i, ds_i^2) at x conjugates left translation by g to right translation by $x^{-1}gx$. Thus D_i is normalized by the symmetries if, and only if, it consists of translation by central elements.

9.9. Corollary. *The compact connected pseudo-riemannian manifolds with consistent absolute parallelism of reductive type are precisely the (M, ϕ, ds^2) of Theorem 9.7 such the both Z/D and $Z\backslash\tilde{M}$ are compact. $Z\backslash\tilde{M}$ is compact if, and only if, each quadric M_i $(u+1 \leq i \leq t)$ is an ordinary 7-sphere $SO(8)/SO(7)$. Z has a discrete subgroup D such that Z/D is compact if, and only if, the 3-form τ of the construction of the Lie algebra $\mathfrak{p}_{-1} = \mathfrak{g}(\tau, \mathfrak{w})$ of M_{-1} can be chosen with rational coefficients.*

Proof. We have a fibration $M = D\backslash\tilde{M} \rightarrow Z\backslash\tilde{M}$ with fibre Z/D. The total space M is compact if, and only if, both fibre Z/D and base $Z\backslash\tilde{M}$ are compact.

$Z\backslash\tilde{M}$ is the product of the $Z_i\backslash M_i$, hence is compact if and only if each $Z_i\backslash M_i$ is compact. If $-1 \leq i \leq u$, then $Z_i\backslash M_i$ is reduced to a point, hence is compact. If $u+1 \leq i \leq t$, then Z_i is finite, so $Z_i\backslash M_i$ is compact if and only if M_i is compact; the latter occurs only for $M_i = SO(8)/SO(7)$.

$\mathfrak{p}_{-1} = \mathfrak{g}(\tau, \mathfrak{w})$ is a nilpotent Lie algebra, and has a basis with rational structure constants if and only if τ can be chosen with rational coefficients. The Lie algebra $\mathfrak{p}_0$ of M_0 is commutative. Now a theorem of Mal'cev [10] says that τ can be chosen rational if, and only if, $M_{-1} \times M_0$ has a discrete subgroup with compact quotient.

Suppose that τ can be chosen rational. Then $M_{-1} \times M_0$ has a discrete subgroup with compact quotient, and gives a left translation group E discrete in $Z_{-1} \times Z_0$ with compact quotient. If $1 \leq i \leq u$ with M_i noncompact, a theorem of Borel [2] provides a discrete subgroup of M_i with compact quotient, and its left translation group is a discrete subgroup $D_i \subset Z_i$ with Z_i/D_i compact. In the other cases Z_i is compact, and we take $D_i = \{1\}$. Then $D = E \times D_1 \times \cdots \times D_t$ is a discrete subgroup of Z with Z/D compact.

Conversely let $D \subset Z$ be a discrete subgroup with Z/D compact. Permute the M_i, $1 \leq i \leq u$, so that M_i is noncompact for $1 \leq i \leq v$ and compact for $v+1 \leq i \leq u$. As $Z_{v+1} \times \cdots \times Z_t$ is compact, we replace D with its projection to $Z' = Z_{-1} \times Z_0 \times \cdots \times Z_v$. Now Z' is a simply connected Lie group whose solvable radical is the nilpotent group $Z_{-1} \times Z_0$ and whose semisimple part $Z_1 \times \cdots \times Z_v$ has no compact factor. Thus a theorem of L. Auslander [1] says that $(Z_{-1} \times Z_0)/\{D \cap (Z_{-1} \times Z_0)\}$ is compact, so τ may be chosen with rational coefficients.

9.10. Corollary. *Let $(\tilde{M}, \tilde{\phi}, d\sigma^2)$ be a complete simply connected pseudo-riemannian manifold with consistent absolute parallelism. Then the following conditions are equivalent.*

(i) *$\tilde{\phi}$ is of reductive type relative to $d\sigma^2$, and $(\tilde{M}, \tilde{\phi}, d\sigma^2)$ has a compact globally symmetric quotient (M, ϕ, ds^2).*

(ii) *$\tilde{\phi}$ is of reductive type relative to $d\sigma^2$ and, in the notation of Theorem 9.7,*

(a) *M_{-1} is reduced to a point,*

(b) *if* $1 \leq i \leq u$, *the group* M_i *is compact,*

(c) *if* $u + 1 \leq i \leq t$, *the quadric* M_i *is a 7-sphere.*

(iii) *There is a riemannian metric* $d\rho^2$ *on* $\tilde{M}$ *consistent with* $\tilde{\phi}$. *Then, if* (M, ϕ, ds^2) *is a quotient of* $(\tilde{M}, \tilde{\phi}, d\sigma^2)$, $d\rho^2$ *induces a riemannian metric* dr^2 *on* M *consistent with* ϕ.

Proof. Assume (i) and let $(M, \phi, ds^2) = D\backslash(\tilde{M}, \tilde{\phi}, d\sigma^2)$. Let D_i be the projection of D to Z_i. If $-1 \leq i \leq u$, then D_i is central in Z_i by Corollary 9.8, and Z_i/D_i is compact by Corollary 9.9. That proves (a) and (b) of (ii); (c) follows directly from Corollary 9.9. Thus (i) implies (ii). For the converse let D be a lattice in M_0.

Assume (ii). Let dr_0^2 be any translation-invariant riemannian metric on M_0. For $1 \leq i \leq u$ let dr_i^2 be the metric induced by the negative of the Killing form of $\mathfrak{p}_i$. For $u + 1 \leq i \leq t$ let dr_i^2 be the usual riemannian metric of constant curvature. Now $d\rho^2 = dr_0^2 \times \cdots \times dr_t^2$ has the required properties. Thus (ii) implies (iii). Corollary 9.3 provides the converse.

10. Appendix: Lie triple systems

We collect the basic facts on Lie triple systems.

A. Foundations: N. Jacobson's work ([7], *or* [8])

A *Lie triple system* (LTS) is a vector space $\mathfrak{m}$ with a trilinear "multiplication" map

$$\mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m} \quad \text{denoted} \quad (x, y, z) \mapsto [x\, y\, z]$$

such that

$$[x\,x\,z] = 0 = [x\,y\,z] + [z\,x\,y] + [y\,z\,x]\ , \tag{10.1a}$$

$$[ab[x\,y\,z]] = [[a\,b\,x]yz] + [[b\,a\,y]xz] + [xy[a\,b\,z]]\ . \tag{10,1b}$$

If $\mathfrak{l}$ is a Lie algebra and $\mathfrak{m} \subset \mathfrak{l}$ is a subspace such that $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}$, then $\mathfrak{m}$ is a LTS under the composition $[x\,y\,z] = [[x, y], z]$; for then (10.1a) is anticommutative and the Jacobi identity, and (10.1b) follows by iteration of the Jacobi identity.

Let $\mathfrak{m}$ be a LTS. By *derivation* of $\mathfrak{m}$ we mean a linear map $\delta \colon \mathfrak{m} \to \mathfrak{m}$ such that

$$\delta([x\,y\,z]) = [\delta(x)\,y\,z] + [x\,\delta(y)\,z] + [x\,y\,\delta(z)]\ . \tag{10.2a}$$

We denote

(10.2b) $\mathfrak{d}(\mathfrak{m})$: the Lie algebra of derivations of $\mathfrak{m}$.

If $\{a_i\}, \{b_i\} \subset \mathfrak{m}$, we have the derivations $\sum \delta_{a_i, b_i}$ where $\delta_{a,b}(x) = [a\, b\, x]$ for $a, b, x \in \mathfrak{m}$. Derivations of that sort are *inner derivations*. Denote

(10.2c) $\mathfrak{d}_0(\mathfrak{m})$: ideal in $\mathfrak{d}(\mathfrak{m})$ consisting of inner derivations.

Now consider the vector space

(10.3a) $\mathfrak{h}(\mathfrak{m}) = \mathfrak{d}(\mathfrak{m}) + \mathfrak{m}$ vector space direct sum

with the algebra structure

(10.3b) $[D + x, E + y] = ([D, E] + \delta_{x,y}) + (D(y) - E(x))$.

Then $\mathfrak{h}(\mathfrak{m})$ is a Lie algebra, called the *holomorph* of $\mathfrak{m}$ because every derivation of $\mathfrak{m}$ is the restriction of an inner derivation of $\mathfrak{h}(\mathfrak{m})$. Also, $\mathfrak{d}_0(\mathfrak{m}) = [\mathfrak{m}, \mathfrak{m}]$ inside $\mathfrak{h}(\mathfrak{m})$, so the Lie subalgebra of $\mathfrak{h}(\mathfrak{m})$ generated by $\mathfrak{m}$ is the *standard Lie enveloping algebra* of $\mathfrak{m}$:

(10.3c) $\mathfrak{l}_s(\mathfrak{m}) = \mathfrak{d}_0(\mathfrak{m}) + \mathfrak{m}$ vector space direct sum.

Let $\mathfrak{m}$ and $\mathfrak{n}$ be LTS. If $f: \mathfrak{m} \to \mathfrak{n}$ is a linear map such that

$$f[x\, y\, z] = [f(x)\, f(x)\, f(z)] ,$$

then f is a *homomorphism*. If f is one-one and onto, i.e., if $f^{-1}: \mathfrak{n} \to \mathfrak{m}$ exists, then f^{-1} is a homomorphism and f is an *isomorphism*. If $\mathfrak{l}$ is a Lie algebra and $f: \mathfrak{m} \to \mathfrak{l}$ is an injective LTS homomorphism such that $f(\mathfrak{m})$ generates $\mathfrak{l}$, then we say that $\mathfrak{l}$ or $(\mathfrak{l}, f)$ is a *Lie enveloping algebra* of $\mathfrak{m}$. Those always exist, for one has $\mathfrak{l}_s(\mathfrak{m})$.

The usual tensor algebra method provides a Lie enveloping algebra $\mathfrak{l}_U(\mathfrak{m})$ with the property: *if* $(\mathfrak{l}, f)$ *is any Lie enveloping algebra of* $\mathfrak{m}$, *then* f *extends to a Lie algebra homomorphism of* $\mathfrak{l}_U(\mathfrak{m})$ *onto* $\mathfrak{l}$. Thus $\mathfrak{l}_U(\mathfrak{m})$ is called the *universal Lie enveloping algebra* of $\mathfrak{m}$. The case $\mathfrak{l} = \mathfrak{l}_s(\mathfrak{m})$ shows

$$\mathfrak{l}_U(\mathfrak{m}) = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m} \qquad \text{vector space direct sum.}$$

Also, if $n = \dim \mathfrak{m}$ then $\dim \mathfrak{l}_U(\mathfrak{m}) < n(n+1)/2$.

Let $\mathfrak{m}$ be a LTS. By *subsystem* of $\mathfrak{m}$ we mean a subspace $\mathfrak{k} \subset \mathfrak{m}$ such that $[\mathfrak{k}\, \mathfrak{k}\, \mathfrak{k}] \subset \mathfrak{k}$. By *ideal* in $\mathfrak{m}$ we mean a subspace $\mathfrak{i} \subset \mathfrak{m}$ such that $[\mathfrak{i}\, \mathfrak{m}\, \mathfrak{m}] \subset \mathfrak{i}$ (and thus also $[\mathfrak{m}\, \mathfrak{m}\, \mathfrak{i}] \subset \mathfrak{i}$). The ideals of $\mathfrak{m}$ are just the kernels $f^{-1}(0)$ of LTS homomorphisms $f: \mathfrak{m} \to \mathfrak{n}$, $\mathfrak{n}$ variable; if $\mathfrak{i}$ is an ideal then $\mathfrak{m}/\mathfrak{i}$ inherits a LTS structure from $\mathfrak{m}$, the projection $p: \mathfrak{m} \to \mathfrak{m}/\mathfrak{i}$ is a homomorphism, and $\mathfrak{i} = p^{-1}(0)$ kernel.

B. *Structure*: *W. G. Lister's work* [9]

Let $\mathfrak{m} \subset \mathfrak{l}$ be a LTS in Lie enveloping algebra. Then $[\mathfrak{m}, \mathfrak{m}]$ and $[\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$ are subalgebras of $\mathfrak{l}$, so $\mathfrak{l} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$. If $[\mathfrak{m}, \mathfrak{m}] \cap \mathfrak{m} = 0$, then one verifies that $\mathfrak{l}$ has an automorphism σ whose $+1$ eigenspace is $[\mathfrak{m}, \mathfrak{m}]$ and whose -1 eigenspace is $\mathfrak{m}$. This applies in particular to $\mathfrak{l}_s(\mathfrak{m})$ and to $\mathfrak{l}_U(\mathfrak{m})$, and it is the basic connection between LTS theory and symmetric space theory.

The *derived series* of a LTS $\mathfrak{m}$ is the chain

(10.4a) $$\mathfrak{m} = \mathfrak{m}^{(0)} \supset \mathfrak{m}^{(1)} \supset \cdots \supset \mathfrak{m}^{(k)} \supset \cdots$$

of ideals of $\mathfrak{m}$ defined by

(10.4b) $$\mathfrak{m}^{(k+1)} = [\mathfrak{m}\ \mathfrak{m}^{(k)}\ \mathfrak{m}^{(k)}] .$$

$\mathfrak{m}$ is *solvable* if its derived series terminates in 0, i.e., if some $\mathfrak{m}^{(k)} = 0$. If $\mathfrak{m}$ is solvable, then every Lie enveloping algebra of $\mathfrak{m}$ is a solvable Lie algebra.

The *radical* of $\mathfrak{m}$ is the span of the solvable ideals of $\mathfrak{m}$; it is the maximal solvable ideal in $\mathfrak{m}$, and we denote

(10.5a) $$\mathfrak{r}(\mathfrak{m}):\quad \text{radical of } \mathfrak{m}.$$

If $\mathfrak{r}(\mathfrak{m}) = 0$, then $\mathfrak{m}$ is *semisimple*. In general there is a Levi decomposition

(10.5b) $$\mathfrak{m} = \mathfrak{s} + \mathfrak{r}(\mathfrak{m}) , \quad \mathfrak{s} \text{ semisimple}, \quad \mathfrak{s} \cap \mathfrak{r}(\mathfrak{m}) = 0 .$$

The projection $\mathfrak{m} \to \mathfrak{m}/\mathfrak{r}(\mathfrak{m})$ maps $\mathfrak{s} \cong \mathfrak{m}/\mathfrak{r}(\mathfrak{m})$.

If $\mathfrak{m}$ has no proper ideals, then $\mathfrak{m}$ is *simple*. If $[\mathfrak{m}\ \mathfrak{m}\ \mathfrak{m}] = 0$, then $\mathfrak{m}$ is *commutative*. If $\mathfrak{m}$ is simple, then either it is semisimple and noncommutative, or it is 1-dimensional and commutative.

If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are LTS, then their *direct sum* is the LTS $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ given by

$$[x_1 + x_2 \quad y_1 + y_2 \quad z_1 + z_2] = [x_1y_1z_1] + [x_2y_2z_2];\ x_i, y_i, z_i \in \mathfrak{m}_i .$$

Note that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are complementary ideals in $\mathfrak{m}$. Conversely, if $\mathfrak{m}$ is a LTS with complementary ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, then $\mathfrak{m} \cong \mathfrak{m}_1 \oplus \mathfrak{m}_2$.

If $\mathfrak{m}$ is semisimple, then $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_t$ where the $\mathfrak{m}_i$ are its distinct simple ideals; thus $\mathfrak{m}^{(1)} = \mathfrak{m}$, every derivation of $\mathfrak{m}$ is inner, and every linear representation of $\mathfrak{m}$ is completely reducible. Conversely, if $\{\mathfrak{m}_1, \cdots, \mathfrak{m}_t\}$ are noncommutative simple LTS, then $\mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_t$ is semisimple.

The structure of semisimple LTS was just reduced to that of simple LTS. For the latter, let $\mathfrak{m} \subset \mathfrak{l}_U(\mathfrak{m})$ be a noncommutative simple LTS in its universal Lie enveloping algebra. Then there are just two cases, as follows.

(10.6) If $\mathfrak{m}$ is the LTS of a (necessarily simple) Lie algebra $\mathfrak{k}$, then $\mathfrak{l}_U(\mathfrak{m}) = \mathfrak{k} \oplus \mathfrak{k}$ in such a manner that

$$\mathfrak{m} = \{(x, -x) : x \in \mathfrak{k}\} \quad \text{and} \quad [\mathfrak{m}, \mathfrak{m}] = \{(x, x) : x \in \mathfrak{k}\} .$$

Thus $\mathfrak{m}$ is the -1 eigenspace of the involutive automorphism $(x, y) \mapsto (y, x)$ of $\mathfrak{l}_U(\mathfrak{m})$.

(10.7) If $\mathfrak{m}$ is not the LTS of a Lie algebra, then $\mathfrak{l}_U(\mathfrak{m})$ is simple, and $\mathfrak{m}$ is the -1 eigenspace of an involutive automorphism of $\mathfrak{l}_U(\mathfrak{m})$.

Now the classification of simple LTS over an algebraically closed field is more or less identical to the classification of compact irreducible riemannian symmetric spaces.

Let $\mathfrak{m}$ be a LTS. Then the *center* of $\mathfrak{m}$ is

(10.8) $$\mathfrak{z}(\mathfrak{m}) = \{x \in \mathfrak{m} : [x\, \mathfrak{m}\, \mathfrak{m}] = 0\} .$$

The representation theory of $\mathfrak{m}$ coincides with that of $\mathfrak{l}_U(\mathfrak{m})$. Thus the following conditions are equivalent.

(10.9a) $\mathfrak{m}$ has a faithful completely reducible linear representation.

(10.9b) $\mathfrak{l}_U(\mathfrak{m})$ has a faithful completely reducible linear representation, i.e., $\mathfrak{l}_U(\mathfrak{m})$ is "reductive".

(10.9c) $\mathfrak{l}_U(\mathfrak{m}) = \mathfrak{z} \oplus \mathfrak{s}$ where $\mathfrak{z}$ is its center, $\mathfrak{s}$ is semisimple, and $\mathfrak{s} = [\mathfrak{l}_U(\mathfrak{m}), \mathfrak{l}_U(\mathfrak{m})]$ derived algebra.

(10.9d) $\mathfrak{m} = \mathfrak{z}(\mathfrak{m}) \oplus \mathfrak{m}^{(1)}$, and the derived LTS $\mathfrak{m}^{(1)} = [\mathfrak{m}\, \mathfrak{m}\, \mathfrak{m}]$ is semisimple.

Under the equivalent conditions (10.9) we say that $\mathfrak{m}$ is *reductive*. From the corresponding Lie algebra situation, we say that a subsystem $\mathfrak{n} \subset \mathfrak{m}$ is *reductive in* $\mathfrak{m}$ if the adjoint representation of $\mathfrak{l}_U(\mathfrak{m})$ restricts to a completely reducible representation of $\mathfrak{n}$. Thus

(10,10a) $\mathfrak{m}$ is reductive $\Leftrightarrow$ $\mathfrak{m}$ is reductive in $\mathfrak{m}$,

(10.10b) if $\mathfrak{m}$ is reductive, and $\mathfrak{n}$ is reductive in $\mathfrak{m}$, then $\{x \in \mathfrak{m} : [x\, \mathfrak{n}\mathfrak{n}] = 0\}$ is reductive in $\mathfrak{m}$.

C. *Invariant bilinear forms*

Now we introduce a notion of invariant bilinear form for LTS. That is the key to application of the theory of reductive LTS to the theory of pseudo-riemannian symmetric spaces.

Let $\mathfrak{l}$ be a Lie algebra. Recall that *invariant bilinear form* on $\mathfrak{l}$ means a symmetric bilinear form b on $\mathfrak{l}$ such that $b([x, y], z) = b(x, [y, z])$. It then follows that

$$b(z,[[y,x],w]) = b([[x,y],z],w) = b(x,[[w,z],y]) \ .$$

The main example is the *trace form*

$$b_\pi(x,y) = \text{trace}\ \pi(x)\pi(y)$$

of a linear representation π of $\mathfrak{l}$. The algebra $\mathfrak{l}$ is reductive if, and only if, it has a nondegenerate trace form. However (3.7) shows that a non-reductive algebra might carry a nondegenerate invariant bilinear form.

Let $\mathfrak{m}$ be a LTS. By *invariant bilinear form* on $\mathfrak{m}$ we mean a symmetric bilinear form b such that

$$(10.11) \qquad b(z,[y\ x\ w]) = b([x\ y\ z],w) = b(x,[w\ z\ y]) \ .$$

The preceding discussion shows that the restriction of an invariant bilinear form on a Lie enveloping algebra of $\mathfrak{m}$ is an invariant bilinear form on $\mathfrak{m}$.

10.12. Lemma. *Let* $\mathfrak{m}$ *be a* LTS, *and* b *an invariant bilinear form on* $\mathfrak{m}$.

(i) *The center* $\mathfrak{z} = \{x \in \mathfrak{m} : [x\ \mathfrak{m}\ \mathfrak{m}] = 0\}$ *and the derived system* $\mathfrak{m}^{(1)} = [\mathfrak{m}\ \mathfrak{m}\ \mathfrak{m}]$ *satisfy* $b(\mathfrak{z}, \mathfrak{m}^{(1)}) = 0$.

(ii) *If* $\mathfrak{i}$ *is an ideal in* $\mathfrak{m}$, *then* $\{x \in \mathfrak{m} : b(x,\mathfrak{i}) = 0\}$ *is an ideal in* $\mathfrak{m}$.

(iii) *If* $\mathfrak{l}$ *is a Lie enveloping algebra of* $\mathfrak{m}$ *in which* $[\mathfrak{m},\mathfrak{m}] \cap \mathfrak{m} = 0$, *then* $\mathfrak{l}$ *carries an invariant bilinear form* b' (*in the sense of Lie algebras*) *such that* $b = b'|_{\mathfrak{m}}$.

Proof. For (i) note $b(\mathfrak{z},\mathfrak{m}^{(1)}) = b(\mathfrak{z},[\mathfrak{m}\ \mathfrak{m}\ \mathfrak{m}]) = b([\mathfrak{z}\ \mathfrak{m}\ \mathfrak{m}],\mathfrak{m}) = b(0,\mathfrak{m}) = \{0\}$.

For (ii) let $\mathfrak{j} = \{x \in \mathfrak{m} : b(x,\mathfrak{i}) = 0\}$. It is a linear subspace of $\mathfrak{m}$. If $i \in \mathfrak{i}$, $j \in \mathfrak{j}$ and $x, y \in \mathfrak{m}$, then

$$b([j\ x\ y], i) = b(j,[i\ y\ x]) \in b(\mathfrak{j},\mathfrak{i}) = \{0\} \ ,$$

so $[j\ x\ y] \in \mathfrak{j}$.

For (iii) we define b' on $\mathfrak{m} \times \mathfrak{m}$ to agree with b; *we define* $b'([\mathfrak{m},\mathfrak{m}],\mathfrak{m}) = 0$; and we define b' on $[\mathfrak{m},\mathfrak{m}] \times [\mathfrak{m},\mathfrak{m}]$ by

$$b'([x,y],[z,w]) = b([x\ y\ z],w) \qquad \text{for } x,y,z,w \in \mathfrak{m} \ .$$

That gives us a symmetric bilinear form b' on $\mathfrak{l}$ such that $b = b'|_{\mathfrak{m}}$. Now we check that b' is invariant, i.e., that $b'([p,q],r) = b'(p,[q,r])$ for all $p,q,r \in \mathfrak{l}$. It suffices to assume that each of p,q,r is in $[\mathfrak{m},\mathfrak{m}] \cup \mathfrak{m}$ and go by cases.

If $p,q,r \in \mathfrak{m}$, then $[p,q],[q,r] \in [\mathfrak{m},\mathfrak{m}]$ so $b'([p,q],r) = 0 = b'(p,[q,r])$.

If $p,q \in \mathfrak{m}$ and $r = [z,w]$ with $z,w \in \mathfrak{m}$, then $b'([p,q],r) = b'([p,q],[z,w]) = b([p\ q\ z],w) = b(p,[w\ z\ q]) = b(p,[q,[z,w]]) = b'(p,[q,r])$, which takes care of the case $p,q \in \mathfrak{m}$ and $r \in [\mathfrak{m},\mathfrak{m}]$, and the cases $p,r \in \mathfrak{m}$ and $q \in [\mathfrak{m},\mathfrak{m}]$, and $q,r \in \mathfrak{m}$ and $p \in [\mathfrak{m},\mathfrak{m}]$, follow immediately.

If $p \in \mathfrak{m}$ and $q,r \in [\mathfrak{m},\mathfrak{m}]$, then $[p,q] \in \mathfrak{m}$ so $b'([p,q],r) = 0$, and $[q,r] \in [\mathfrak{m},\mathfrak{m}]$ so $b'(p,[q,r]) = 0$. The cases $q \in \mathfrak{m}$ and $p,r \in [\mathfrak{m},\mathfrak{m}]$, and $r \in \mathfrak{m}$ and $p,q \in [\mathfrak{m},\mathfrak{m}]$, follow similarly.

Finally, let $p = [s, t]$, $q = [x, y]$ and $r = [z, w]$ with $s, t, x, y, z, w \in \mathfrak{m}$. Note $[p, q] + [y, [p, x]] + [x, [y, p]] = 0$ and $[q, r] + [[r, x], y] + [[y, r], x] = 0$. Using the invariance already checked, now

$$\begin{aligned} b'([p, q], r) &= b'([[p, x], y], r) - b'([[p, y], x], r) \\ &= b'([p, x], [y, r]) - b'([p, y], [x, r]) \\ &= b'(p, [x, [y, r]]) - b'(p, [y, [x, r]]) \\ &= b'(p, [q, r]) \ . \end{aligned}$$

q.e.d.

Suppose that $\mathfrak{m}$ is a LTS and b is a nondegenerate invariant bilinear form. Then $x \in \mathfrak{z} \Leftrightarrow b([x\, \mathfrak{m}\, \mathfrak{m}], \mathfrak{m}) = 0 \Leftrightarrow b(x, [\mathfrak{m}\, \mathfrak{m}\, \mathfrak{m}]) = 0$. Thus

(10.13a) $\mathfrak{z}^{\perp} = \mathfrak{m}^{(1)}$ relative to the form b, so

(10.13b) $\dim \mathfrak{m} = \dim \mathfrak{z} + \dim \mathfrak{m}^{(1)}$.

The analogous fact (that $\mathfrak{z}^{\perp} = [\mathfrak{l}, \mathfrak{l}]$) holds for nondegenerate invariant bilinear forms on Lie algebras.

We extend a theorem of Dieudonné from Lie algebras to LTS.

10.14. Proposition. *Let* $\mathfrak{m}$ *be a* LTS, *and* b *a nondegenerate invariant bilinear form on* $\mathfrak{m}$. *If* $\mathfrak{m}$ *has no nonzero ideal* $\mathfrak{i}$ *such that* $[\mathfrak{i}\, \mathfrak{m}\, \mathfrak{i}] = 0$, *then* $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_t$ *where the* $\mathfrak{m}_j$ *are simple ideals,* $b(\mathfrak{m}_j, \mathfrak{m}_k) = 0$ *for* $j \neq k$, *and each* $b|_{\mathfrak{m}_j \times \mathfrak{m}_j}$ *is a nondegenerate invariant bilinear form.*

Proof. Let $\mathfrak{m}_1$ be a minimal ideal in $\mathfrak{m}$. From Lemma 10.12, $\mathfrak{m}_1^{\perp} = \{x \in \mathfrak{m} : b(x, \mathfrak{m}_1) = 0\}$ is an ideal, so also $\mathfrak{i} = \mathfrak{m}_1 \cap \mathfrak{m}_1^{\perp}$ is an ideal. If $i, j \in \mathfrak{i}$ and $x, y \in \mathfrak{m}$, then

$$b([i\, x\, j], y) = b(i, [y\, j\, x]) \in b(\mathfrak{i}, \mathfrak{i}) = \{0\} \ ;$$

so $[\mathfrak{i}\, \mathfrak{m}\, \mathfrak{i}] = 0$ by nondegeneracy of b. Thus $\mathfrak{i} = 0$ by hypothesis. Now $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_1^{\perp}$. The proposition holds for $\mathfrak{m}_1^{\perp}$ by induction on $\dim \mathfrak{m}$. q.e.d.

Conversely, (10.6) and (10.7) show that every semisimple LTS carries a nondegenerate invariant bilinear form, in characteristic zero.

Now with (3.6) and (3.7) in mind, we introduce

10.15. Definition. Let $\mathfrak{m}$ be a LTS, and b a nondegenerate invariant bilinear form on $\mathfrak{m}$. Suppose

(i) b is nondegenerate on the center of $\mathfrak{m}$, and

(ii) if $\mathfrak{i}$ is an ideal in $\mathfrak{m}$ such that $[\mathfrak{i}\, \mathfrak{m}\, \mathfrak{i}] = 0$, then $\mathfrak{i}$ is central in $\mathfrak{m}$, i.e., $[\mathfrak{i}\, \mathfrak{m}\, \mathfrak{m}] = 0$.

Then we say that the pair $(\mathfrak{m}, b)$ is of *reductive type*.

10.16. Theorem. *Let* $\mathfrak{m}$ *be a* LTS, *and* b *a nondegenerate invariant bilinear form on* $\mathfrak{m}$ *such that* $(\mathfrak{m}, b)$ *is of reductive type. Then* $\mathfrak{m}$ *is reductive. Moreover*

(10.17a) $$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_t ,$$

where

(10.17b) $\mathfrak{m}_0$ *is the center of* $\mathfrak{m}$ *and the other* $\mathfrak{m}_i$ *are simple ideals,*

(10.17c) $$b(\mathfrak{m}_i, \mathfrak{m}_j) = 0 \quad \textit{for} \quad i \neq j , \quad \textit{and}$$

(10.17d) *each* $b|_{\mathfrak{m}_i \times \mathfrak{m}_i}$ *is nondegenerate.*

Conversely, if $\mathfrak{m}$ *is a reductive* LTS *over a field of characteristic zero, then it carries a nondegenerate invariant bilinear form* b *such that* $(\mathfrak{m}, b)$ *is of reductive type.*

Proof. Let $(\mathfrak{m}, b)$ be of reductive type, $\mathfrak{m}_0$ be the center of $\mathfrak{m}$, and $\mathfrak{m}' = \{x \in \mathfrak{m} : b(x, \mathfrak{m}_0) = 0\}$. As b is nondegenerate on $\mathfrak{m}_0$, now $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}'$ and $b = b_0 \oplus b'$. Let $\mathfrak{i} \subset \mathfrak{m}'$ be an ideal such that $[\mathfrak{i}\, \mathfrak{m}'\, \mathfrak{i}] = 0$. As $[\mathfrak{i}\, \mathfrak{m}_0\, \mathfrak{i}] \subset [\mathfrak{m}_0\, \mathfrak{m}\, \mathfrak{m}] = 0$, now $[\mathfrak{i}\, \mathfrak{m}\, \mathfrak{i}] = 0$. Thus $\mathfrak{i} \subset \mathfrak{m}_0$, so $\mathfrak{i} = 0$. Now Proposition 10.14 says $\mathfrak{m}' = \mathfrak{m}_1 \oplus \cdots \mathfrak{m}_t$ with $b' = b_1 \oplus \cdots \oplus b_t$. That proves (10.17).

Conversely let $\mathfrak{m}$ be reductive. Then $\mathfrak{m} = \mathfrak{z} \oplus \mathfrak{s}$ where $\mathfrak{z}$ is its center and $\mathfrak{s}$ is semisimple. Let b'' be any nondegenerate bilinear form on $\mathfrak{z}$, and choose a nondegenerate invariant bilinear form b' on $\mathfrak{s}$; then $b = b'' \oplus b'$ is a nondegenerate invariant bilinear form on $\mathfrak{z} \oplus \mathfrak{s} = \mathfrak{m}$ and is nondegenerate on $\mathfrak{z}$. If $\mathfrak{i} \subset \mathfrak{m}$ is an ideal with $[\mathfrak{i}\, \mathfrak{m}\, \mathfrak{i}] = 0$, then $[\mathfrak{i}\, \mathfrak{i}\, \mathfrak{i}] = 0$, so $\mathfrak{i}$ is solvable, whence $\mathfrak{i} \subset \mathfrak{z}$.

10.18. Corollary. *Let* $\mathfrak{m}$ *be a reductive* LTS, *and* b *a nondegenerate invariant bilinear form on* $\mathfrak{m}$. *Then* $(\mathfrak{m}, b)$ *is of reductive type, the center* $\mathfrak{m}_0$ *of* $\mathfrak{m}$ *is b-orthogonal to the derived system* $\mathfrak{m}^{(1)}$, *and the distinct simple ideals of* $\mathfrak{m}^{(1)}$ *are mutually b-orthogonal.*

Proof. As $\mathfrak{m}$ is reductive, $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}^{(1)}$, and (10.13a) says $b(\mathfrak{m}_0, \mathfrak{m}^{(1)}) = 0$. Now apply Proposition 10.14 to the semisimple system $\mathfrak{m}^{(1)}$.

10.19. Corollary. *Let* $\mathfrak{l}$ *be a Lie algebra over a field of characteristic zero. Then* $\mathfrak{l}$ *is reductive if, and only if,*

(i) *every abelian ideal of* $\mathfrak{l}$ *is central, and*

(ii) $\mathfrak{l}$ *has a nondegenerate invariant bilinear form which is nondegenerate on the center of* $\mathfrak{l}$.

If $\mathfrak{l}$ *is reductive and b is a nondegenerate invariant bilinear form, then the center z of* $\mathfrak{l}$ *is b-orthogonal to the derived algebra* $\mathfrak{l}'$, *and the distinct simple ideals of* $\mathfrak{l}'$ *are mutually b-orthogonal.*

Conditions (i) and (ii) both fail for the algebra (3.7).

Condition (i) does not imply (ii), as seen from the Lie algebra $\mathfrak{l}$ of $Sp(n, R) \cdot H_n$ where H_n is the $(2n + 1)$-dimensional Heisenberg group, $Sp(n, R)$ acts irreducibly on a ($2n$-dimensional) complement to the center Z of H_n, and $Sp(n, R)$ acts trivially on Z. Here $\mathfrak{z}$ is the only abelian ideal in $\mathfrak{l}$.

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